Journal of Mathematics and Music
Vol. 03, No. 01, February 2009, 1–26

Auto Similar Melodies

Emmanuel AMIOT

1 rue du Centre, F 66570 St NAZAIRE, France

(*v1.0.0 released july 2007*)

The present work surged from purely musical topics, namely the notion of 'selfRep melody' as defined by composer Tom Johnson in [7], and reaped interesting mathematical rewards as well as musical. It is about melodies that contain themselves in augmentation, and some generalizations thereof. Some of the mathematical by-results are new to the best of the author's knowledge. Applications range from melodies to rhythms, and include some new results on mosaics. Finally, extension to approximate and 'non invertible' autosimilar melodies will prove that this notion is both widespread and universal.

Notations: $\mathbb{Z}_n$ *stands for the cyclic set with* n *elements, considered as the group* $(\mathbb{Z}_n, +)$ *or as a ring as the occasion demands – or just a set.*
We denote the greatest common divisor of a, n *by* $\gcd(a, n)$*. Quite often, calculations make sense both in* $\mathbb{Z}_n$ *and* $\mathbb{Z}$*. When in doubt, consider that* $a \in \mathbb{Z}_n$ *can be identified with the smallest non negative integer in* $\mathbb{Z}$ *with residue* a*.*
Division is sometimes denoted by $|$*: for instance* $8 \mid 4 \mod 12$ *as* $4 = 5 \times 8 \mod 12$*.*
The invertible elements of $(\mathbb{Z}_n, \times)$ *are the generators of the additive group* $(\mathbb{Z}_n, +)$*; they form a multiplicative group,* $\mathbb{Z}_n^*$*.*
Any set might be given by the list of its elements: $(0, 3, 5)$*; or by some defining property, e.g.* $\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n,\ \gcd(a, n) = 1\}$*.*
The subgroup generated by some element g *of a group* G *is denoted by* $\mathrm{gr}(g)$*. For instance* $\mathrm{gr}(a) = (\mathbb{Z}_n, +) \iff a \in \mathbb{Z}_n^*$*.*
A periodic melody M *is a map from* $\mathbb{Z}_n$ *into some musical space, usually pitches or notes, or equivalently a periodic sequence:* $\forall k \in \mathbb{Z}, M_{k+n} = M_k$*. So* M_k *is well defined for* $k \in \mathbb{Z}_n$*.*
The affine group modulo n *is* Aff_n*, the set of affine bijections in* $\mathbb{Z}_n$*, i.e.* $x \mapsto a\,x + b$ *for* $(a, b) \in \mathbb{Z}_n^* \times \mathbb{Z}_n$*.*
The ***order*** *of an element* g *of a group* G *is the cardinality of* $\mathrm{gr}(g)$*, i.e. the smallest integer* $r > 0$ *with* $g^r = e$*, the unit element of group* G*. It is classically characterized by the following equivalence:*

$$g^k = e \iff o(g) \text{ is a divisor of } k$$

The cardinality of any set G *is denoted* $|G|$*: e.g.* $o(a) = |\,\mathrm{gr}(a)|$*.*

Introduction

Symmetries of $\mathbb{Z}_n$ have been well explored under the group of translations and inversion (T/I), for instance in American Set Theory. Such transformations are expressed by maps $x \mapsto \pm x + b$. But despite the obvious interest of more general affine transforms in $\mathbb{Z}_n$, e.g. $x \mapsto a\,x + b$, very little research has been made on orbits of affine maps, or subgroups of the affine group[1]. This is mildly surprising, as many interesting notions are invariant under affine transformations: interval content (up to permutation), all-interval sets, limited transposition modes[2], tiling (mosaic) property, series and all-interval series, to name but a few.

Professor in Class Preps, Perpignan, France. Email: manu.amiot@free.fr

[1] Except for $c = 12$ and WITH loss of generality, as vehemently stigmatised in [10][section 11.5.4.2]. See nonetheless [8] or [9] for the case $n = 12$.

[2] This one was brought to our attention by M. Andreatta. It is easy to prove, when one recalls that any affine map permutes the interval vector.

Journal of Mathematics and Music
ISSN 1745-9737 print / ISSN 1745-9745 online © 2009 Taylor & Francis Ltd.
http://www.tandf.co.uk/journals
DOI: 10.1080/17459730xxxxxxxxx

The present paper is on the one hand a mathematical study of orbits of affine maps operating on a cyclic group. On the other hand it is developing a musically interesting notion of autosimilarity, just like famous fractals (the Cantor Set, Koch flake, Sierpinski sponge) but discrete. Diverse musical renderings are possible; in the simplest a melody plays within itself contrapunctically, something the Cantor of Leipzig might have dreamed of. Several instances of such melodies have been identified in classical music.

1 First definitions, historical examples.

We begin with the simplest case, when all augmentations begin on the same note. This is historically the case studied by Tom Johnson in [7], though he came across the more general case, with different starting points, which will be studied in section III; other generalizations will occur in the last sections.

1.1 *Autosimilar melody with ratio a.*

DEFINITION 1.1 *Let M be a* **periodic** *melody with period n: $M_0, \dots M_n = M_0, M_{n+1} = M_1, \dots$ wherein the values M_k are musical events (pitch classes for instance) and k is some measure of time. M is* ***autosimilar***[1] *with ratio a iff*

$$\forall k \in \mathbb{Z}_n \quad M_{a\,k} = M_k$$

This means that taking one note every a beats lets hear the same melody, only slower; or equivalently that some augmentation of the melody is part of the melody itself, as is quite obvious on the score below (with $a = 3$). This explains why the melody has to be infinite. Non-periodic solutions are possible[2], but this would be another subject.

1.2 *Musical examples*

Figure 1. First bars of 'la Vie est si courte'

[1]We decided to change Tom Johnson's 'selfRep' to 'autoSimilar', that he himself used in the broad sense of 'product of some iterated process', because it is really the traditional mathematical meaning, for instance with classical fractals.

[2]like motif xx enlarged, e.g. ...dcbaabcd... obtained from aa, b..b, c....c and so on.

Of course, use of augmentation is quite ancient. J.S. Bach is probably the best known exponent of melodies played simultaneously with their augmentations in numerous fugas; he is also famous for having several voices heard inside one monody (*Suites* for solo strings spring to mind), but we haven't found yet striking examples of his using both ideas at the same time. Tom Johnson has discovered this possibility around 1980, and is probably the first composer who made use of it consciolsly and extensively, as in *la Vie Est Si Courte* above, or *Loops for Orchestra* below, fig. 18.

But his works are by no means the earliest featuring autosimilar melodies: another famous american piece where the melody is autosimilar (with ratio 4) is Glen Miller's *In the Mood.* This is perfectly audible as the extraction of one note out of four coincides with strong beats, and probably quite voluntary on Miller's part as he studied with the mathematically minded (some say 'obsessed') Joseph Schillinger[1].

Figure 2. Thema of 'in the Mood'

But it might surprise many readers to realise that much more ancient western music features autosimilarity: it is found in Beethoven's Fifth Symphony, arguably unconsciously as the ratio 3 autosimilarity is not outlined (fig. 5); or, more interestingly, the ubiquitous Alberti Bass, well known from for instance the beginning of Mozart's Sonata in C major K. 545, is an excellent example, with autosimilarities at ratios 3, 5 and generally every odd number. In Mozart's first bar, the right hand significantly offers a choice of the three notes repeated by the left hand, at a ratio approximating 3 (three notes for eight beats).

Figure 3. Alberti Bass with augmentation by 3

1.3 *Mathematical generation.*

THEOREM 1.2 *Any autosimilar melody with ratio a and period n is built up from orbits of the affine map $x \mapsto a \times x \mod n$: if the orbit of x is defined as*

$$\mathcal{O}_x = \{a^k x \mod n, k \in \mathbb{Z}\} = a^{\mathbb{Z}}.x,$$

then for each note p of the melody, the subset of indexes $M^{-1}(p) = \{i \in \mathbb{Z}_n, M_i = p\}$ is one such orbit, or an union of several ones.

Proof It is sufficient to prove that if $M_x = p$ then $M_k = p$ for all $k \in \mathcal{O}_x$, hence every orbit will come *in toto* for a given note. But this is obvious from the definition, as

$$M_k = M_{a^m \times x} = M_{a^{m-1} \times x} = \dots M_x = p$$

[1]This was pointed out by T. Johnson. Almost forgotten nowadays, Schillinger taught a number of prominent composers before WWII. To quote Henry Cowell in the preface of his book on Schillinger, "The idea behind the Schillinger System is simple and inevitable: it undertakes the application of mathematical logic to all the materials of music and to their functions, so that the student may know the unifying principles behind these functions, may grasp the method of analyzing and synthesizing any musical materials that he may find anywhere or may discover for himself, and may perceive how to develop new materials as he feels the need for them."

□

A well known fact about orbits is worth recalling here, namely that $x \in \mathcal{O}_y \iff y \in \mathcal{O}_x$.

In group theory, these orbits are the classes of the action of the cyclic subgroup generated by the map $f : x \mapsto ax \mod n$.

At this point it seems a good idea to demand that a (or f) indeed generates a subgroup, which means that a is coprime with n or equivalently $a \in \mathbb{Z}_n^*$. As will be seen below, interesting situations arise when this condition is dropped, but this will not happen before section V. So from now on,

$\boxed{\text{The ratio of an autosimilar melody is assumed to be coprime with the period.}}$

Example 1.3 *The* ***abstract*** *melody generated by ratio 3 modulo 8 has five orbits: 0 is sent to 0 so (0) is one orbit. 1 is sent to 3 and 3 sent to 1, so (1 3) is another one. (4), (5 7), (2 6) are the remaining orbits, hence the general structure, $x, y\dots$ denoting arbitrary pitches, is:*

x	y	z	y	t	u	z	$u\dots$
0	1	2	3	4	5	6	$7\dots$

The Alberti Bass (cf. fig. 3) has less than five notes. This comes from using the same note for different orbits: putting arbitrarily (more about this later) C on 0 and 4, G on odd beats i.e. (1 3) and (5 7), and E on (2 6) we have it, autosimilar with ratio 3: ***C*** *G E* ***G*** *C G* ***E*** *G C* ***G*** *...*

Here several orbits have been collapsed on identical notes: $z = E, x = t = C, u = y = G$. Hence it is necessary to define:

Definition 1.4 *A* ***primitive*** *autosimilar melody is a melody generated by a ratio a and a modulo n with different notes for different orbits. In other words, there as many different notes as possible for the given n, a.*

As will be seen below, several mathematical results will only stand for primitive melodies: obviously the possibility of collapsing all the orbits into as little as one (a one note melody, or even a melody of silences... like Cage's 4'33" !) hurts any attempt at a classification of symmetries.

1.4 *The simple case of n prime.*

In this paragraph we assume that the period n is a prime number. This is not really necessary for further study, but it helps come to grips with the notion of autosimilar melodies.

Proposition 1.5 *One orbit has only one note : $\mathcal{O}_0 = (0)$. All others share the same cardinality, which is the multiplicative order of a:*

$$o(a) = |\mathcal{O}_1| = |\{a^k \mod n,\ k \in \mathbb{Z}\}| = \min\{k > 0,\ a^k = 1 \mod n\}$$

Proof The orbit of 1 is exactly the subgroup of ${\mathbb{Z}_n}^*$ generated by a, e.g. all different powers of $a \mod n$: hence $\sharp\mathcal{O}_1 = o(a)$.

Let $x \neq 0$, now the map $y \mapsto y \times x$ is one-to-one (as x must be invertible, $\mathbb{Z}_n$ being a field when n is prime) and maps precisely $\mathcal{O}_1$ onto $\mathcal{O}_x$: hence $|\mathcal{O}_x| = |\mathcal{O}_1| = o(a)$. □

This is the only case where the number of different notes in the melody is easily computed, i.e. $1 + \dfrac{n-1}{o(a)}$.

Musically it is a natural idea to put a rest (or silence) on the singleton 0; so does Tom Johnson in many instances. The result above also proves that such autosimilar melodies are instances of tilings, or mosaics, with augmentation: the tile $\mathcal{O}_1$ will tile the line (e.g. $\mathbb{Z}$) with its augmentations $x_1\mathcal{O}_1, x_2\mathcal{O}_1 \dots$, leaving aside only the multiples of n. An example will clarify this: let $n = 7$ and choose $a \in \mathbb{Z}_n$ with order less than $n - 1$, e.g. $a = 2$: then the orbit of 1 modulo 7 is $(1, 2, 4)$ and the last orbit $\mathcal{O}_2$ is $(3, 6, 12)$ which reduces modulo 7 to $(3, 5, 6)$. Leaving the reduction aside enables to hear the augmentation, like in the following musical rendering (fig. 4 was done in *OpenMusic*) of this *canon con augmentatio*:

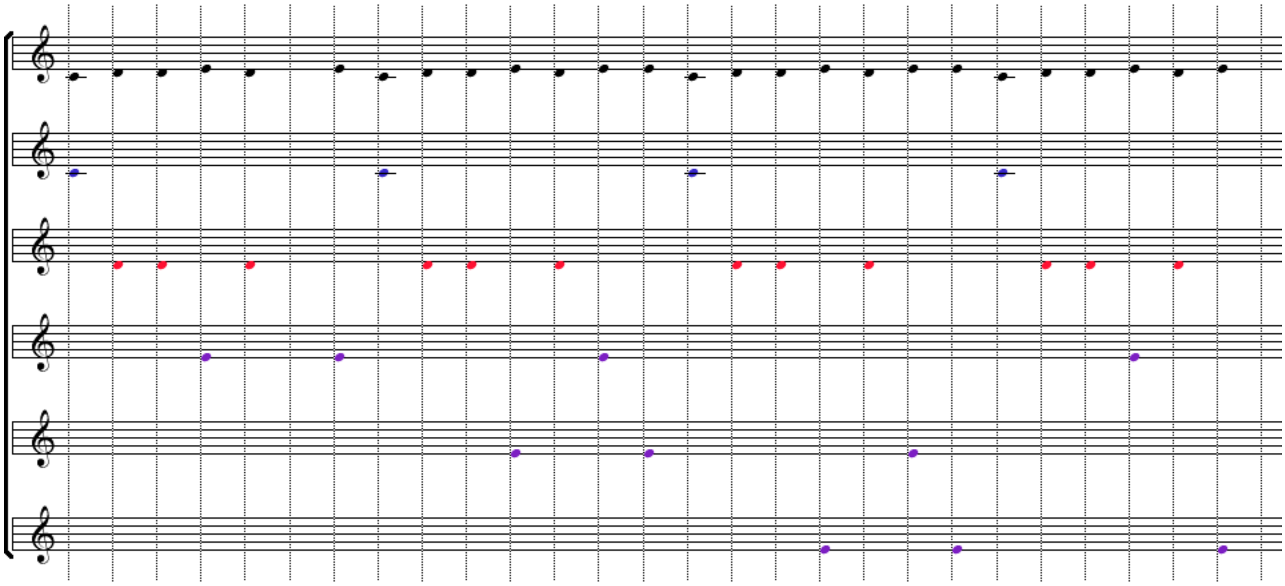

Figure 4. Tiling with augmentation

Tilings by augmentation without holes or overlapping will be tackled in section IV.

Another simple result, given in [5] for n prime but both true and simpler in the general case:

Proposition 1.6 *For any period $n > 1$, the 'melody' $xyyyyyyy\dots xyyyyyy\dots$ is autosimilar*[1] *for any ratio a (coprime with n).*

This is obvious: x stands on the orbit of 0, and the y's are just on the union of all other orbits, whatever the value of a.

Of course, such a melody (with one single note repeated on $n-1$ beats out of n) seems a bit simplistic. It is no longer when one begins to extract its augmentations, as on fig. 5 with $n = 4, a = 3$.

Figure 5. Beethovenian autosimilarity

1.5 *Some figures.*

In the intermediate case of $x \mapsto a\,x \mod n$, we can give a few explicit formulas. As will be proved in the next section, these formulas will still be effective about half of the time in the most general case.

[1] But it is not *primitive* in general.

1.5.1 *Number of occurences of one note.* Some special cases are easy: $\mathcal{O}_0$ always has a single element, and $\mathcal{O}_1$ is the multiplicative subgroup generated by a, i.e. the set of powers of a, and its cardinality is equal to the order $o(a)$ of a. This is exactly what happened for n prime.

The group $\mathbb{Z}_n^*$ however, though abelian, is fairly complicated. In particular, the maximal order of any element $a \in \mathbb{Z}_n^*$, that is to say the largest possible number of occurences of one note in a $n-$periodic primitive autosimilar melody, is given by CARMICHAEL's Λ function[1]. Also, even for simple a's, for instance for $a = p$ prime, the lenghts of orbits are by no mean easy to compute[2]. This hinges on a new behaviour: contrariwise to the n prime case, the map

$$\mathcal{O}_1 \ni a^k \stackrel{t \mapsto x \times t}{\longrightarrow} x \times a^k \in \mathcal{O}_x$$

is no longer one to one.

PROPOSITION 1.7 *The length of any orbit, i.e. the number of occurences of a given note in a primitive autosimilar melody of ratio a and period n, is a **divisor** of $o(a)$.*

Proof It is easy enough to see that the length of any orbit $\mathcal{O}_x$ is at most $o(a)$, as obviously $a^k = 1 \Rightarrow x \times a^k = x$. To get the more precise result we need a little algebra. Consider the epimorphism

$$\begin{aligned} \Psi_x : &\mathbb{Z} \to \mathcal{O}_x \\ &k \mapsto a^k\, x \mod n \end{aligned}$$

$o(a)$ is a **period** of Ψ_x, meaning that $\Psi_x(k + o(a)) = \Psi_x(k)$ for all $k \in \mathbb{Z}$. Now the set of all periods of any map, here

$$\mathcal{T}(\Psi_x) = \{\tau \in \mathbb{Z},\ \forall k \in \mathbb{Z},\ \Psi_x(k + \tau) = \Psi_x(k)$$

is an additive group. But subgroups of $\mathbb{Z}$ are monogenous: $\mathcal{T}(\Psi_x) = r\,\mathbb{Z}$. As $o(a) \in \mathcal{T}(\Psi_x)$ we get that r divides $o(a)$, and r, smallest period of Ψ_x, is clearly the length of $\mathcal{O}_x$ (in short, f acts on $\mathcal{O}_x$ as a circular permutation). $\square$

It is useful to visualise all this orbits as little clocks of different sizes, ticking at different speeds, with at least one great clock whose size is a multiple of all others. Each multiplication by $a \mod n$ ticks every clock, and after a whole 'day', e.g. after $o(a)$ ticks, all clocks must have resumed their initial positions. This is the substance of the above proposition. The perception of an autosimilar melody is then explained as an aural illusion: multiplying by a, i.e. picking one note every a beat, does completely rearrange the order of all the notes inside, but as the different onsets of a given note belong to the same orbit, we believe we hear the same note at the same moment.

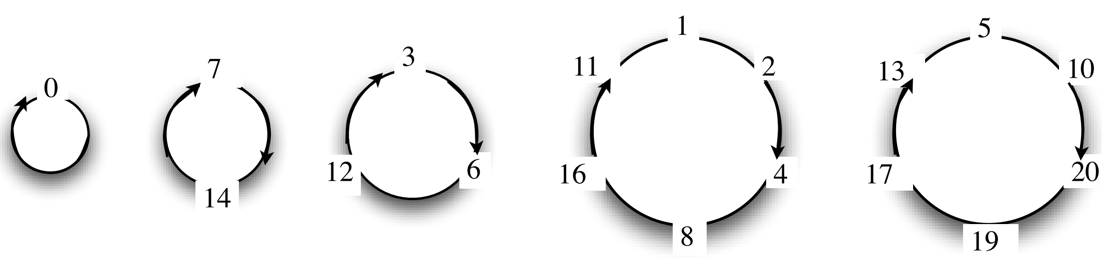

Figure 6. Several clocks ticking together: $n = 21, a = 2$

[1]The CARMICHAEL function Λ verifies $\Lambda(p^\alpha q^\beta \dots) = \mathrm{lcm}[\Lambda(p^\alpha), \Lambda(q^\beta), \dots]$ with $\Lambda(p^\alpha) = \Phi(p^\alpha) = (p-1)p^{\alpha-1}$ except when $p = 2 < \alpha$. See http://mathworld.wolfram.com/CarmichaelFunction.html for details.

[2]The orbits $(x, p\,x, p^2 x \dots)$ are called cyclotomic orbits, their lengths are the degrees of the irreducible factors of $X^n - 1$ in the ring of polynomials on the field with p elements, see [1] for a musical application to a species of rhythmic canons.

This result looks a little unsatisfactory, as one would like to predict the length of the orbit of a given x. A precise answer requires, not algebra, but arithmetics:

Proposition 1.8 *Let $d = \gcd(x, n)$; the length of $\mathcal{O}_x = \{x, a\,x, a^2\,x, \dots\}$ is the order of a modulo n/d, i.e. the smallest integer $k > 0$ with $a^k = 1 \mod n/d$. In particular, $\mathcal{O}_x$ is of maximal length whenever x and n are coprime.*

Proof The nicest way to see this is perhaps to see $\mathcal{O}_x$ as an orbit of the map $y \mapsto a\,y$ operating on $x\,\mathbb{Z}_n$, the set of all multiples of x; but this subset is a cyclic group, precisely the subgroup generated by $d = \gcd(x, n)$ in $\mathbb{Z}_n$, with n/d elements. Dividing by d all elements of the orbit yields the orbit of $y = x/d$ in the cyclic group $\mathbb{Z}_{n/d} = \mathbb{Z}_m$. As $y = x/d$ is now coprime with $m = n/d$, it only remains to prove this special case:

Lemma 1.9 *If $\gcd(y, m) = 1$ then the length of the orbit of y in $\mathbb{Z}_m$ is exactly the order of a modulo m.*

Which is obvious by direct calculation: the orbit loops to its starting point when

$$a^r y = y \iff (a^r - 1)y = 0 \iff a^r - 1 = 0 \iff r \text{ is a multiple of } o(a)$$

as by assumption, y is invertible modulo m and hence simplifiable. More quickly, one could argue as above that the map Ψ_y is one-to-one and maps $\mathcal{O}_1$ to $\mathcal{O}_y$. □

Example 1.10 *Let $n = 15, a = 2$. Then $o(a) = 4$. One singleton orbit, $\mathcal{O}_0$. As 1 and 7 are invertible, their orbits have 4 elements each. So is $\mathcal{O}_3 = (3, 6, 9, 12)$ though 3 is* **not** *invertible mod. 15. This is because $(2^k - 1) \neq 0 \mod 5 = 15/3\ \forall\, k < 4$, hence the orbit still hits maximum size. But $\mathcal{O}_5$ has only 2 elements, as $(2^2 - 1) = 0 \mod 15/5 = 3$.*

We draw the reader's attention to the fact that the automorphisms of the **additive** group $(\mathbb{Z}_n, +)$, which are the $x \mapsto b\,x$ with $b \in {\mathbb{Z}_n}^*$, **permute** these maximal orbits. Indeed they permute *all* orbits, preserving their sizes.

On the other hand, more general $x \mapsto b\,x$ with $\gcd(b, n) > 1$ will not be one-to-one, but none the less carry one orbit into (part of) another; the cases are manifold.

1.5.2 ***Number of single notes.*** A last particular case is that of one note orbits, aka single notes, aka fixed points of the map $x \mapsto a\,x$.

$$\mathcal{O}_x = \{x\} \iff a\,x = x \iff (a - 1)x = 0 \mod n$$

This means exactly that x contains the prime factors of n that are missing in $a-1$. For instance, say $a = 4$ and $n = 15$: such x's are simply the multiples of 5. Hence

Proposition 1.11 *The number of single notes in a primitive autosimilar melody of ratio a and period n is $\gcd(a - 1, n)$. They are the multiples of $n/\gcd(a - 1, n)$.*

This will come as a special case of Prop. 2.15.

Looking for an autosimilar melody with ratio 3 and period 8, we can thus predict $\gcd(3 - 1, 8) = 2$ singletons, and indeed 0 and 4 are the only fixed points of the map $x \mapsto 3x \mod 8$ (both are note C in the Mozart example). This result will be deeply extended in the following section.

Tom Johnson conjectured that the total number of different notes in a melody with period n is at most $3n/4$, as indeed happens for the following melody ($n = 8, a = 5$, 6 different notes):

This is true, as the largest number of different notes is obviously achieved with as many one-note orbits as possible, the rest being organized in two-notes orbits. But the greatest possible value of $\gcd(n, a - 1)$ is $n/2$ and it happens for $a - 1 = n/2$. So n must be even, the fixed points being the even numbers as

$$a \times 2x = (n + 2) \times x = 2\,x \mod n$$

Figure 7. Melody with many different notes

For an odd number,

$$f(2x+1) = a \times (2x+1) = (n+2) \times x + a = 2\,x + 1 + n/2 \mod n$$

and this will yield two-notes orbits whenever this result is still odd, i.e. when $n/2$ is even. Hence

Proposition 1.12 *The maximal number of different orbits is* $3n/4$. *It is achieved when* $n = 4k, a = 2k+1$.

2 Extension to general affine automorphisms

The more general case, especially from an aural point of view, is the action of **any** affine automorphism. Practically, the following definition means that by extracting one note every a notes in the melody, one hears the same melody, though perhaps **with a different starting point** (namely b) – but this is surely irrelevant mathematically, as periodic melodies do not have a starting point (musically this is a different story of course, what with strong and weak beats, orchestration and so on). Consider the following popular rhythmic beat, which is autosimilar with ratio 3 and offset 1:

Figure 8. Autosimilarity with offset

Definition 2.1 *Let M be a periodic melody with period n. M is autosimilar with ratio a and offset b iff symbolically $a\,M + b = M$, i.e.*

$$\forall k \in \mathbb{Z}_n \quad M_{a\,k+b} = M_k$$

Theorem 2.2 *Any autosimilar melody of ratio a, period n, and offset b is built up from orbits of the affine map $x \mapsto a \times x + b \mod n$.*

The proof is identical to that of Thm. 1.2[1].

Remark 1 This new, more general setting, includes the case $a = 1$ with melodies invariant under $x \mapsto x + \tau$, i.e. maps with a period smaller than n. Each orbit, and hence each preimage

$$M^{-1}(p) = \{k, M_k = p\}$$

[1] The mathematical view would be here to define an action of Aff_n on the set of maps $M : \mathbb{Z}_n \to (pcs)$ by $f[M] = M \circ f$. We will make use loosely of this formalism in writing $a\,M + b$ for the sequence $k \mapsto M_{a\,k+b}$.

is then a Limited Transposition Subset of $\mathbb{Z}_n$. We will henceforth drop this case and assume $a1$.

Remark 2 The setting of affine maps modulo n might be unfamiliar to many readers, and a few reminders may be useful. The main point is to distinguish between the **monoid** of general affine maps, and the **group** of affine **transformations**, which are one-to-one maps; these last are exactly the $x \mapsto a\,x + b \mod n$ with $\gcd(a, n) = 1$. Their group Aff_n is a semi-direct product of its translation subgroup (all $x \mapsto x + b$), isomorphic to the group $\mathbb{Z}_n$, and its homotheties subgroup (all $x \mapsto a\,x, \gcd(a, n) = 1$) which is isomorphic to $\mathbb{Z}_n^*$; Aff_n is not abelian and several open problems remain about its structure. [10] is quite right in demanding that the exact sequence

$$0 \to (\mathbb{Z}_n, +) \to (\mathrm{Aff}_n, \circ) \to (\mathbb{Z}_n^*, \times) \to 1 \tag{1}$$

(which is another way of expressing the semi-direct product structure of Aff_n we mentioned) be taken into account; but it is not sufficient to explain all that we will encounter.

For instance, the *Kientsy Loops*[1] melody G F E D E F G D G F E D E F G D. . . can be viewed as generated by $x \mapsto 3x + 6 \mod 8$ if origin is G=0:

0	1	2	3	4	5	6	7	...
6	1	4	7	2	5	0	3	...
G	*F*	*E*	*D*	*E*	*F*	*G*	*D*	...

True, it can be viewed more simply as generated by $x \mapsto 3\,x$ if we decide that the consecutive F, E stand on positions 0,1. This ambivalence will be elucidated in a little while.

2.1 *Orbit lengths*

LEMMA 2.3 *The order of map $f : x \mapsto a\,x + b \mod n$ (e.g. the size of the subgroup $\mathrm{gr}(f)$ generated by f in Aff_n) is given by*

$$o(f) = \min r > 0 \textit{ such that } \gcd(a-1, b) \times (1 + a + \dots a^{r-1}) = 0 \mod n \tag{2}$$

Proof By easy induction,

$$f^k(x) = f \circ f \circ \dots f(x) = a^k\,x + b \times (1 + a + \dots a^{r-1}) \tag{3}$$

If this is equal to x, for all x, then by substracting between x and $x+1$ one gets condition $a^k - 1 = 0$; and putting in $x = 0$ yields $b \times (1 + a + \dots a^{r-1}) = 0$. As $a^k - 1 = (a-1) \times (1 + a + \dots a^{r-1})$ and (by BEZOUT's identity) there exists some u, v with $ub + v(a-1) = \gcd(a-1, b)$, this means that the condition of the lemma is satisified.

Conversely, if it is satisified, as $\gcd(a-1, b)$ divides both $a-1$ and b, then $a^k - 1 = 0$ and $b \times (1 + a + \dots a^{r-1}) = 0$ which yields exactly $f^k(x) = x \ \forall x \in \mathbb{Z}_n$. □

In practice, compute the 'missing factor' $mf = n/\gcd(a-1, b, n)$, and look up the first number $1 + a + \dots a^{r-1}$ that is a multiple of mf.

PROPOSITION 2.4 *$o(f)$ is a* ***multiple*** *of $o(a)$, and a* ***divisor*** *of the smallest integer s satisfying*

$$1 + a + a^2 + \dots a^{s-1} = 0$$

[1]Reference to CD and / or score missing here.

We leave the proof as an exercise (consider the periods of sequences f^k, a^k, and $1 + a + a^2 + \dots a^{k-1}$, or alternatively the sequence 1).

All these numbers are identical for instance when $a - 1$ is coprime with n, as then

$$1 + a + a^2 + \dots a^{s-1} = 0 \iff (1-a)(1 + a + a^2 + \dots a^{s-1}) = 0 \iff 1 - a^s = 0 \iff a^s = 1$$

A technical result allows to get a little more precise:

PROPOSITION 2.5 *Let τ be the number of translations in $\mathrm{gr}(f)$ where $f : x \mapsto a\,x + b$: in practice, one computes τ by iterating f until $f^k(x) = x + k$ [this implies that each orbit, and hence the melody, is $k-$periodic]: then $\tau = n/\gcd(k, n)$ and $o(f) = o(a) \times \tau$.*

Proof Straightforward from the exact short sequence

$$0 \longrightarrow k\,\mathbb{Z}_n \approx \mathbb{Z}_\tau \approx \mathrm{gr}(x \mapsto x + k) \longrightarrow \mathrm{gr}(f) \stackrel{(x \mapsto a\,x+b) \to a}{\longrightarrow} \mathrm{gr}(a) \subset \mathbb{Z}_n^* \longrightarrow 1$$

which mimicks the sequence (1). □

This means heuristically that if a has many ($o(a)$) different powers, then the melody will have few (τ) periods.

Example 2.6 *: consider map $f : x \mapsto 3\,x + 1 \mod 8$. $f \circ f(x) = 3(3\,x + 1) + 1 = 9x + 4 = x + 4 \mod 8$ hence $f^4(x) = (x + 4) + 4 = x + 8 = x \mod 8$ and $o(f) = 4$, as predicted:*

$$\gcd(a - 1, b) = \gcd(3 - 1, 1) = 1 \qquad \tau = 2 \qquad 1 + a = 4, \quad 1 + a + a^2 = 5, \quad 1 + a + a^2 + a^3 = 0 \mod 8$$

On the other hand, with $g : x \mapsto 3\,x + 2 \mod 8$, though $g^2 = id$, strangely g admits no fixed point.

As in section I, we get

PROPOSITION 2.7 *The length of any orbit is a divisor of $o(f)$.*

The metaphor of clocks (see fig. 6) is the same as in the case $b = 0$: all smaller clocks must have looped to initial position when a 'day' is spent, that is to say when f has been iterated a number of times that equals it to identity[1].

This means that the number of occurences of a given note divides $o(f)$ – **but this stands only if the melody is primitive**: random reunions of orbits would put of course this result into shambles. Surprisingly, one result from the easy case $b = 0$ (and even n prime) is still valid:

2.2 *Orbits with maximal length.*

THEOREM 2.8 *There exists at least one orbit with length exactly $o(f)$.*

It entails that $o(f) \leq n$. This[2] is NOT obvious, as

- The group of all affine automorphisms has cardinality $n \times \Phi(n)$ and is non commutative[3]; a subgroup (like the one generated by f) of such a finite group might well have more than n elements.
- In general, the order of a general permutation of n objects may be much greater than n: french composer J. BARRAQUÉ made us of this when interpreting any permutation of 12 objects as a twelve tone row, and generating up to 60 series from a single one in this way[4]. The general formula is that the order of

[1]This is a special case of a general result in group action theory: cardinals of orbits of a finite group divide the cardinality of the group, and more precisely the quotient is the cardinality of the subgroup fixing some given element of the orbit. Here this subgroup is $\mathrm{gr}(f)$.

[2]$o(f) = n$ for instance when $f(x) = x + b, b \in \mathbb{Z}_n^*$. These maps are conjugates in Aff_n of the basic translation $x \mapsto x + 1$. It happens also, surprisingly, for non translations, like $x \mapsto 5\,x + 1 \mod 8$ or $x \mapsto 16\,x + b \mod 45, \gcd(b, 45) = 1$.

[3]except for $n = 2$; here Φ denotes Euler's totient function

[4]A permutation with cycles (= orbits) of lengths 3, 4 and 5 has order 60.

a map that permutes n objects is the lowest common multiple of the cardinalities of its orbits, which is usually more than the greatest among these cardinalities.

Proof We need DIRICHLET's famous theorem on arithmetical sequences:

$$\boxed{\text{In any sequence } \{v, u+v, \dots u\,x+v, \dots\} \text{ with } \gcd(u,v)=1, \text{ there exists an infinity of prime numbers.}}$$

Let $\alpha = (a-1) \wedge b$ and $u = \dfrac{a-1}{\alpha}$, $v = \dfrac{b}{\alpha}$: u, v are coprime integers, hence choosing some large x_0 prime in the sequence as above, which implies coprimality with n [if x_0 is large enough it will not be a prime factor of n] we have:

LEMMA 2.9 *There exists x_0 with $u\,x_0 + v = \dfrac{a-1}{\alpha}x_0 + \dfrac{b}{\alpha}$ invertible (modulo n).*

We then compute the length of $\mathcal{O}_{x_0}$:

$$f^k(x_0) = x_0 \iff \big((a-1)x_0+b\big)(1+a+\dots a^{k-1}) = 0 \iff (u\,x_0+v)\alpha(1+a+\dots a^{k-1}) = 0 \iff \alpha(1+a+\dots a^{k-1}) = 0$$

As seen in Lemma 2.3, this implies that $o(f)$ is a divisor of k, which completes the proof. □

Example 2.10 $n = 10, a = 3, b = 4$.
We compute $\alpha = \gcd(a-1, b) = 2$. Also $r = o(f) = 4$ as $\alpha \times (1+3+9+27) = 0 \mod 10$. And indeed there is one orbit of length 4, $\mathcal{O}_0 = (0, 4, 6, 2)$. Other orbits like $\mathcal{O}_3 = (3)$ or $\mathcal{O}_1$ have lengths that divide 4.

2.3 *Surprising lengths.*

Example 2.11 *: It is remarkable that the length of an orbit can appear unconnected with the order of a: the addition of b in the map $f : x \mapsto a\,x + b$ changes the behaviour of f tremendously. For instance, though $7^2 = 1 \mod 12$, the orbits of $x \mapsto 7\,x + 2 \mod 12$ have length 3 or 6 (this last being the order of f).*

On the other hand, not all divisors of $o(f)$ are lengths of orbits, just as not any divisor of the order of a group corresponds to a subgroup. Still a composer might wish to repeat some note a given number of times, and so try and find the appropriate map $f \in \mathrm{Aff}_n$. It may appear surprising, for instance, that there exists orbits of length 5 in 22-periodic melodies. But this is a corollary to a well-known lemma attributed to CAUCHY: as the whole group Aff_n is of order $n \times \Phi(n)$,

PROPOSITION 2.12 (from Cauchy's Lemma)*Let p be any prime factor of n or $\Phi(n)$, there is an element of $f \in \mathrm{Aff}_n$ with order $o(f) = p$.*

The interest is of course that unexpected, new prime factors crop up in $\Phi(n)$. This is particularly interesting when using general affine transformations because one gets prime factors that do not occur in n. More generally, any result on the order of elements of group Aff_n, e.g. Sylow theorems and the like[1], can be interpreted as a result on lengths of orbits[2].

2.4 *Orbits with one element.*

The shortest orbits are given by fixed points of the affine map. There is a nice geometric characterization:

[1] Composer Tom Johnson suggested that these textbook theorems on finite groups be applied to the context of autosimilar melodies

[2] For instance, if $n = 2^{v(2)}p^{v(p)}q^{v(q)}\dots$, it could be shown that there exists an element of $\mathbb{Z}_n^*$ with order $p^{v(p)}q^{v(q)}\dots$ times some power of 2 (usually $2^{v(2)-2}$), or any divisor of this. This maximal value is called $\Lambda(n)$, see note above on Carmichael's function.

2.4.1 *Fixed points mean homotheties.*

THEOREM 2.13 *Any autosimilar melodie with ratio a and offset b admitting at least one lone note is generated by an homothety $x \mapsto a \times x \mod n$ for some a – if this lone note is chosen as the origin 0 on $\mathbb{Z}_n$. In algebraic terms this means that the map is a* ***conjugate*** *of an homothety.*

This means that, at least musically, more than half of all autosimilar melodies belong to the simple case we studied in the first place. We have already observed this behaviour with the melody of *Kientsy Loops* above.

Proof We have seen that $x \mapsto a\,x \mod n$ admits at least one fixed point – 0. Conversely, let us assume that z is a fixed point of map $x \mapsto a\,x + b$: $a\,z + b = z$. Consider now offsetting the starting point by z, i.e. putting $y = x - z$: hence the new map is g, with

$$f(x) = a\,x + b \iff g(y) = f(x) - z = a\,x + b - z = a\,(y + z) - z = a\,y$$

With this new origin we have reduced f to the simpler case of section 1. □

Geometrically this is obvious, if one is willing to convey his or hers intuition of affine maps into $\mathbb{Z}_n$: if f fixes $z \in \mathbb{Z}_n$, then as f is affine it is completely determined by the value of $f(z+1) = f(z) + a$. But the homothety $h_{z,a}$ with center z and ratio a gives the same values in $z, z+1$, so it IS f. There is also a numerical test, that the reader may check with a direct computation:

THEOREM 2.14 *f is an homothety up to a change of origin if and only iff $f(x) = a\,x + b$ with b a multiple of $a-1$* ***in*** *$\mathbb{Z}_n$ (if b is to be read as a true integer in $\mathbb{N}$, this reads as "b must be a multiple of $\gcd(a-1, n)$").*

It is worthy of note that

- If f is an homothety (i.e. admits some fixed point) then $o(f) = o(a)$, but
- the converse is not true: map $x \mapsto 3\,x + 1 \mod 10$ has no fixed point at all but is of order 4, just the same as its ratio: $3^4 = 81 = 1 \mod 10$ and $3(3\,x+1)+1 = 9\,x + 4 = 4 - x \mod 10, 4 - (4 - x) = x$. Two orbits have length 4, and the other has length 2.

2.4.2 *Number of fixed points.*

Contrarily to our intuition in planar geometry, an affine map mod n may well have several fixed points, e.g. 'centers'.

PROPOSITION 2.15 *Let $d = \gcd(a-1, n)$: if $d \mid b$ (in $\mathbb{N}$) then we get d fixed points. Else there is none.*

Proof x_0 is a fixed point $\iff (a-1)x_0 = b \mod n$.
If $d = 1$, then $a - 1$ is invertible, and $x_0 = (a-1)^{-1}b$ is the only possible fixed point.
If d divides $a - 1$ and n but not b, we get an impossibility.
Assume d does divide $b : b = k\,d$: the equation now reads

$$(a-1)x_0 = b \mod n \iff \frac{a-1}{d}x_0 = \frac{b}{d} \mod \frac{n}{d}$$

and as $\dfrac{a-1}{d}$ and $\dfrac{n}{d}$ are coprime, we get one solution modulo n/d, that is to say d solutions modulo n. □

(for example : $x \mapsto 3x + 2k + 1, n = 2p$ has no fixed point).
So an homothety might have different centers, that is to say an autosimilar melody can be related to the simplest case in more ways than one. Musically this allows to show the autosimilarity on different circular permutations of the initial melody.

2.4.3 *Number of homotheties.*

From theorem 2.14 above we can easily compute the number of homotheties in Aff_n, as it is a simple matter to enumerate all acceptable b's for a given a, which are all multiples of $\gcd(n, a-1)$:

PROPOSITION 2.16 *The number of homotheties in* Aff_n*, identity map excluded, is given by formula*

$$N_{hom}(n) = \sum_{\substack{2 \le a \le n-1 \\ \gcd(a,n)=1}} \frac{n}{\gcd(n, a-1)}$$

Its maximum value is achieved when n is prime, as for all values of a, $\gcd(n, a-1) = 1$ and hence $N_{hom}(n) = n(n-2) = (n-1)^2 - 1$, among $n^2 - n$ affine maps. By contrast, $N_{hom}(30) = 63$ only; and almost one affine map out of 3 is a homothety when n is a power of 2. It seems that 4,6 and 12 are the only values of n with $N_{hom}(n) < n$. The proportion of homotheties among general affine maps, depending much on the factorisation of n, is pretty erratic:

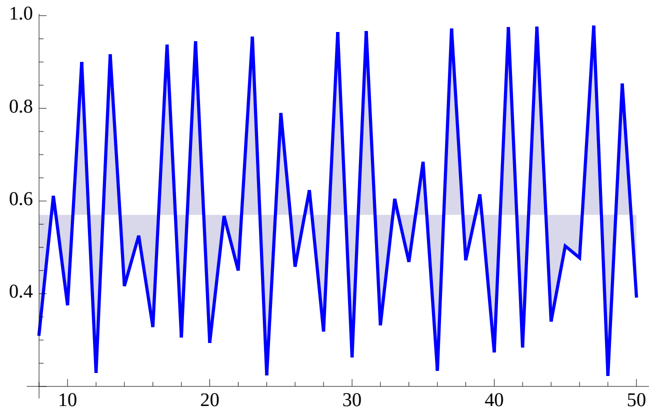

Figure 9. proportion of homotheties in Aff_n

On average and for practical purposes, the proportion of homotheties in Aff_n is around 57% for $3 \le n \le 200$.

2.5 *Number of different notes*

The question of the maximal possible number of different notes (that is to say the number of orbits) was solved in the simpler case $b = 0$ with Prop. 1.12. Now in a way, the identity map $x \mapsto x$ makes up an 'autosimilar melody' with n different fixed points, but this is musically meaningless, and even mathematically a bit extreme. The following result states that the simpler case is also the general one. We leave the proof to the reader.

THEOREM 2.17 *The maximum number of different notes for an autosimilar melody with period* n *is* $3n/4$*, which is reached exactly when* $n = 4k, a = 2k+1$ *and* b *is* 0 *or* $n/2$.

In general, the total number of notes (or orbites) varies wildly with the modulo and ratio. There is a formula, making use of stabilizer groups and the celebrated 'Lemma that is not Burnside's ' of group theory and combinatorics, but it is more efficient computationally just to compute all orbits and enumerate them. Here is this formula, wherein $X(g)$ is the number of fixed points of $g \in \mathrm{Aff}_n$ (computed from Prop. 2.15):

PROPOSITION 2.18 *Let* $r = o(f)$*,* $d_k = \gcd(a^k - 1, n)$ *and* $X(f^k) = \begin{cases} d_k & \text{if } d_k \mid b(1 + a + \dots a^{k-1}) \\ 0 & \text{if not} \end{cases}$.

The total number of notes, i.e. of orbits under the action of f*, i.e. under action of the group* $G = \mathrm{gr}(f) \subset$

Aff_n, *is*

$$\sum_{g\in G} X_g/|G| = \frac{1}{r}\sum_{k=1}^{r} X(f^k)$$

Here is a plot with the mean value of the number of different notes for any given n, mean taken on all ratios $a > 1$ coprime with n and all possible offsets b.

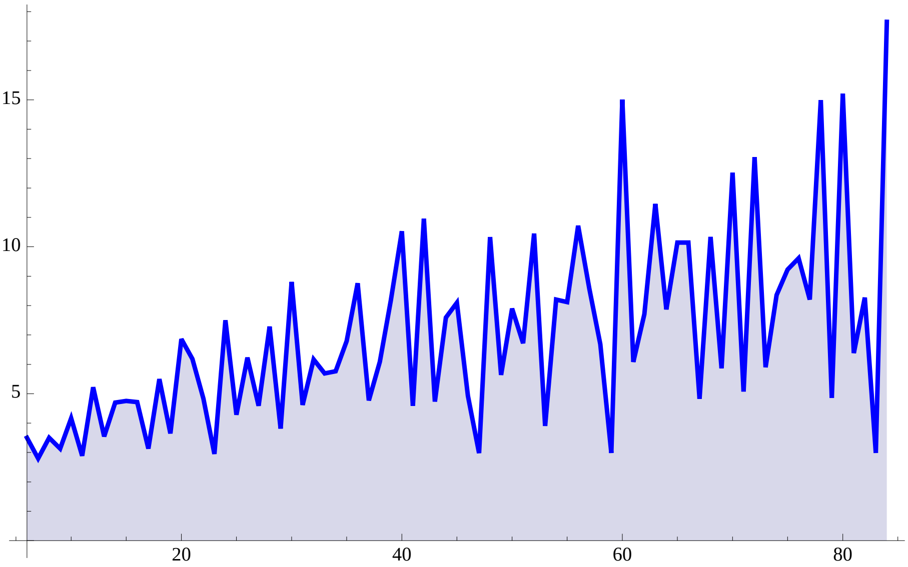

Figure 10. Average number of notes

These computations enable to compute a fairly reasonable[1] value of the probability for a melody to be [primitive] autosimilar, namely the number of partitions of $\mathbb{Z}_n$ into affine orbits, over the number of all partitions (e.g. 2^n). This probability decreases quickly, for $n = 20$ it is $p = 0.000084877$ and for $n = 72$, with only 480 partitions into affine orbits, the probability is negligible ($\approx 10^{-19}$). Whatever the propriety of this mode of calculation of a probability, it shows that autosimilar melodies are highly organised material, and that autosimilarity is a significant feature indeed.

3 Other symmetries

We will remain in the more general context of affine automorphisms $x \mapsto a\,x + b$, not only homotheties.

3.1 *Symmetry group*

Definition 3.1 *The symmetry group of a (periodic) melody M is the subgroup of Aff_n containing all maps g satisfying $g[M] = M$, that is to say $\forall k\ M_{g(k)} = M_k$. One says that g* ***stabilizes*** *M.*

Two extreme examples are a melody that is **not** autosimilar, meaning its symmetry group contains only the identity map; and the melody with only one note, which has the whole group Aff_n for symmetries.

The Alberti bass C G E G C G E G ... admits **all** odd ratios for autosimilarity, and more precisely its symmetry group is made of eight distinct maps mod 8 (check this is an abelian group):

$$x \mapsto x, 3x, 5x, 7x, x+4, 3x+4, 5x+4, 7x+4$$

[1]There are many different ways to define a probability space on melodies, and about as many different probability values.

As any autosimilar melody is built up from some map $f \in \mathrm{Aff}_n$, it is obvious that any f^k stabilises the melody. Indeed this means exactly that the melody IS autosimilar under map f. A partial reciprocal is true:

THEOREM 3.2 *Let M be a primitive autosimilar melody generated by map $f : x \mapsto a\,x$. Then any homothety $g \in \mathrm{Aff}_n$, e.g. $g(x) = c\,x$ that stabilises M, is a power of f, e.g. $\exists k\ c = a^k$.*

*When c is no power of a, g **permutes** the orbits, that is to say stabilises the **rhythmic structure** of the melody, while exchanging its notes.*

Proof Assume $g(x) = c\,x$ stabilises M. In particular, the orbit $\mathcal{O}_1$ which contains powers of a is globally invariant under g, meaning $g(1) = c \in \mathcal{O}_1$ is some power of a, qed.

If c is *not* a power of a, as we have seen already, maps of the kind $x \mapsto c\,x$ turn $\mathcal{O}_x$ into $\mathcal{O}_{c\,x}$. □

This means, quite significantly, that in considering only the simpler affine maps (homotheties), only the obvious symmetries will occur. The picture is of course different in the whole affine group, and we do not have a general result. Of course, nothing can be said when the melody is not primitive as collapsing some orbits together will increase the symmetry group. Apart from the Alberti Bass, we quote below (fig. 18) one page of the score of *Loops for Orchestra* by Tom Johnson, composed by Tom Johnson, wherein the melody admits several different ratios.

A redeeming feature is the fact that many (more than half) affine maps are indeed homotheties, up to a change of origin. In this case the result is simple:

THEOREM 3.3 *If $f : x \mapsto a\,x + b$ is a homothety (from Thm 2.14, this means that* $\gcd(a-1, n)$ *divides b), then the symmetry group of any primitive autosimilar melody built from f is just the group of powers of f.*

Proof Let us assume, up to a change of origin, that 0 is a fixed point of f. Then (0) is an orbit, stable under any element of the symmetry group. This means that all these maps are of the form $x \mapsto c\,x$ and by the last theorem they must be powers of f. □

Remark 1 One could also look for the larger subgroup of Aff_n preserving the **set structure** of orbits – meaning exchanges of notes are allowed. In the case of the theorem above we fall back on the whole group of homotheties, isomorphic to $\mathbb{Z}_n^*$.

In the more general case, the situation can be pretty complicated: for instance the map $x \mapsto 3x + 1 \mod 8$ (cf. Fig. 8) builds up the melody CCGGCCGGCCGGCCGGCCGGCCGG... which admits as many symmetries as the Alberti bass:

$$x \mapsto x, x+4, 3x+1, 3x+5, 5x, 5x+4, 7x+1, 7x+5$$

Complicated symmetry groups are possible (this last one, sometimes called H_8, is **not** abelian). As all powers of f are in the symmetry group, we can at least predict that

LEMMA 3.4 *The order of the group of symmetries of M is a multiple of $o(f)$.*

Proof $o(f)$ is the order of a (cyclic) subgroup of the symmetry group, hence divides its order by LAGRANGE's theorem. □

It is interesting for composers, and maybe analysts alike, to find a melody with a given symmetry group. This is a kind of 'inverse Galois problem'. For instance one may wish to find an 8-periodic melody, palindromic, and autosimilar with ratio 3. Hence the orbit of 1 must contain $-1 = 7$ (for palindromicity); as it contains 3, $5 = 7 \times 3$ is there too. Hence $\mathcal{O}_1$ contains all odd numbers.

Acting similarly with the remaining residue classes, one finds the primitive solution $\mathcal{O}_1 = (1, 3, 5, 7), \mathcal{O}_2 = (2, 6)$ with 4 and 8 standing alone as fixed points. One rendering of this unique solution is none other than the Alberti Bass. This leads to the most general definition so far, which indeed should be the starting point for the study of melodies invariant under some affine maps[1]:

DEFINITION 3.5 *An autosimilar melody M with period n and symmetry group $G \subset \mathrm{Aff}_n$ is any map $M : \mathbb{Z}_n \to$ (some pitch set) that satisfies*

$$\forall f \in G \quad \forall k \; M_{f(k)} = M_k$$

THEOREM 3.6 *Such a melody is built up from the orbits of G, i.e. each pitch appears on indexes that are a reunion of orbits $\mathcal{O}_x = \{f(x), f \in G\}$.*

Algorithmically speaking, one usually wishes to consider only some symmetries in G. An orbit will be produced by repeatedly applying the given affine maps, starting with a given seed, until no new number is produced[1]. This last definition reduces to the above one when the group of symmetries is cyclic, generated by just one map.

It must be pointed out that the group of symmetries eventually achieved is usually larger than the group one starts from; not any odd group is the symmetry group of some object: for instance modulo 8, this Klein group

$$K = \{x \mapsto x, x \mapsto 3x + 1, x \mapsto x + 4, x \mapsto 3x + 5\}$$

is not the symmetry group of a melody: its orbits are

$$(0, 1, 4, 5) \qquad (2, 3, 6, 7)$$

and the symmetry group of any melody built up from these [say C C G G C C G G. . .] is strictly larger (for instance it contains $x \mapsto 5x + 4$), with 8 elements.

Remark 2 Musically speaking, for such an autosimilar melody with a non cyclic group of affine symmetries, it is possible to extract the melody from itself at different ratios, as can be seen on fig. 11 made in OpenMusic. A similar situation will arise in the last section.

3.2 *Palindroms*

Now we can clarify when a given autosimilar melody will be a palindrom, as this means exactly that $x \mapsto -x$ is an element of the group G of symmetries. Here is an example, in figure 12. At least this is crystal clear when G is generated by $h_a : x \mapsto a\,x$, as we have seen that the symmetries must be powers of h_a:

THEOREM 3.7 *M, primitive autosimilar melody with ratio a, period n and offset 0, will be palindromic $\iff$ some power of a is equal to $-1 \mod n$.*

The question of whether -1 is a **square** modulo n (a quadratic residue) is pretty well known, but the concept of -1 being a **power** residue appears to be novel. The sequence of moduli n admitting such a possibility: 5, 7, 9, 10, 11, 13, 14, 17, 18, 19, 21, 22, 23, 25, 26, 27, 28, 29, 31, 33, 34, 35, 36, 37, 38, 39. . . has been added to Sloane's online encyclopedy of integer sequences under number A126949[2]. It is a rather common occurence.

[1] The implementation of the construction of such a melody has been made available to composers and other musicians in the software *OpenMusic*, [2].

[1] The underlying idea is that: any orbit is a fixed point of the action of any set of generators of G on the set of all subsets of $\mathbb{Z}_n$. This was implemented in *OpenMusic*, cf. [2].

[2] http://www.research.att.com/ njas/sequences/A126949

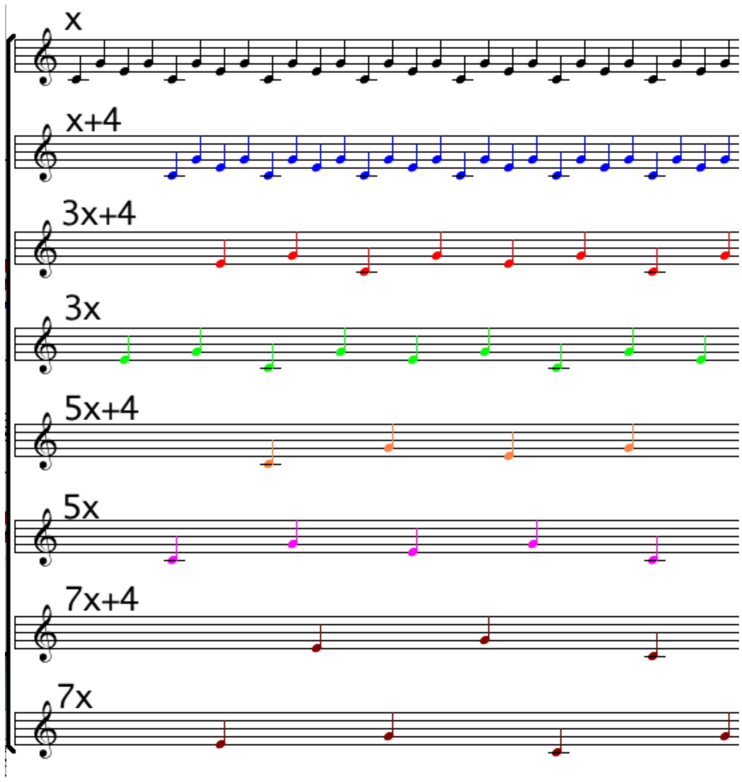

Figure 11. Different augmentations of the Alberti bass

Figure 12. Autosimilar palindrom with period 14

For instance mod 25, $2, 3, 4, 7, 8, 9, 12, 13, 14, 17, 18, 19, 22, 23$ all have some power equal to -1; on the other hand, there are no such powers modulo 6, 8, 12, 15, 16, 20, 24, 30, 32…

Again, this stands only for *primitive* melodies. Of course, it is always possible to build a palindromic (autosimilar) melody from *any* autosimilar melody by just collapsing together notes belonging to orbits that are symmetrical (the map $x \mapsto -x$ exchanges orbits of a primitive autosimilar melody with ratio a and offset 0). For example see the original and the palindromized on figure13.

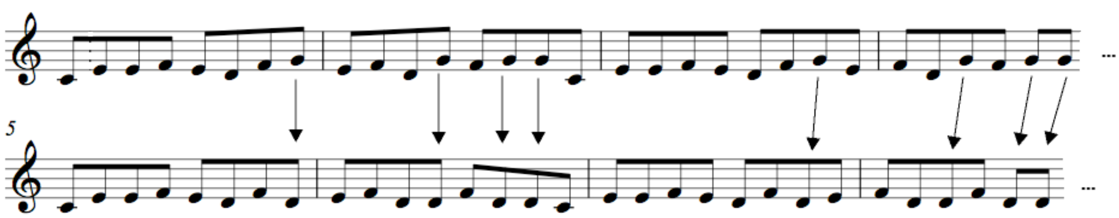

Figure 13. a melody and its palindrom deformation

What of a melody autosimilar with offset ? For it to be an exact palindrom, the condition above is still necessary, but not always sufficient as seen from this example: the map $x \mapsto 3x + 1 \mod 8$ generates orbits $(0, 1, 4, 5), (2, 7, 6, 3)$ and melody A A B B A A B B … which is not, strictly speaking, a palindrom, though it becomes one by allowing an offset of the origin. This is general:

Theorem 3.8 *Let M, primitive autosimilar melody generated by $f : x \mapsto ax + b \mod n$; if some power* *of a is equal to $-1 \mod n$, then M will be palindromic* ***up to some offset****.*

Proof Assume that $a^k = -1$. Then $f^k(x) = -x + c$ for some c and hence $M = f^k[M] = -M + c$, i.e. M is palindromic with a different starting point. □

The reciprocal, unfortunately, is false when the map is not an homothety. For instance, no power of 3 equals -1 modulo 8, though $x \mapsto 3x + 1$ generates a palindromic melody with offset. Still for some pairs (a, n), there are no b's at all that will allow $x \mapsto a\,x + b$ to generate a palindrom, e.g. $x \mapsto 8x + b \mod 15$.

3.3 *Combined symmetries*

The disposition of pitches can occur in a space where symmetries are possible (e.g., inversion).

3.3.1 ***A conjecture.*** The last paragraphs enable to clarify a conjecture from Johnson, which quite uncharacteristically happened to be wrong (and was proved so in [5]). This was the conjecture ([7]):

> A related melody produced by playing a melodic loop [= a periodic melody] at some ratio other than 1:1, can never be the inversion of the original loop, unless it is also a retrograde of the original loop.

What Tom means by 'related melody' is just some $f[M] = (M_{f(k)})_{k \in \mathbb{Z}_n}$; an autosimilar melody is precisely a melody M that is equal to one of its related melodies.

Here we are looking for periodic melodies satisfying a condition

$$M_{f(k)} = p - M_k \quad \forall k \in \mathbb{Z}_n \qquad\qquad \text{(where } p \text{ is some constant)}$$

For this to happen, we need the musical space wherein M takes its values to possess some minimal algebraic structure, which is usually true in most models.

The conjecture states that the above condition implies

$$\forall k\ M_{-k} = M_{f(k)}. \tag{4}$$

Feldman, who first in history shed a mathematical look on these melodies (he used some himself as a composer) but unfortunately only ever published the short [5] about them, shrewdly points out that Tom's conjecture will be true when n is prime [and f is homothetic] and provides a counterexample with period 15.

The inversion condition implies that M itself is autosimilar under f^2, as

$$\forall k \quad M_{f^2(k)} = p - M_{f(k)} = p - \big(p - M_k\big) = M_k$$

As the inversion acts on the orbits of f, they must have even cardinality: let us consider the simpler case $f(x) = a\,x$, say 1 is pitch x, then a is pitch $p - x$, a^2 is pitch x again, a.s.o. What we want to avoid in order to disprove the conjecture is $-1 \in \mathcal{O}_1$, as then the melody would be invariant under $x \mapsto -a\,x$. As seen above, -1 is often a power residue (this explains Johnson's error) and, for instance when n is prime, if a is of even order $2k$ then $-1 = a^k$ as $a^k \neq 1$ is the only other solution of $X^2 = 1$ in the field $\mathbb{Z}_n$ (this is Feldman's argument). But in $\mathbb{Z}_{15}$ for instance, $X^2 = 1$ has other solutions (eg 4); taking $a = 2, n = 15$ and filling in the ordered orbits (1 2 4 8), (3 6 12 9), (5 10), (7 14 13 11) with alternate opposite values of one note (0 is C, 1 is C♯, 2 is B, a.s.o.), we get by construction $M_{f(k)} = -M_k\ \forall k$, close to Feldman's example.

3.3.2 ***The ratio that retrogrades.*** This suggests to look also for melodies with one alternate melody being its retrograde. While composing we found (C D E C D F C D G C D E . . .), autosimilar with period 9, ratio 4 and offset 6: picking every odd note turns the melody into its retrograde (up to offsetting): (C E D C G D C F D C E D C . . .), i.e. $2M + 1 = -M + 8$. As it happens, this is a general phenomenon, and it has nothing to do with 4 being a power of 2, quite the reverse:

Figure 14. One note out of two gives inversion, not retrograde

Proposition 3.9 *Let M be an autosimilar melody with ratio a and any offset; put $c = -a^{-1} \mod n$; then picking one note every c yields the retrograde $-M$ (up to some offset).*

This is most easily audible when $c = 2$, i.e. when $a = \dfrac{n-1}{2}$ (for some odd n!)

Proof Symbolically $M = a\,M + b$, hence $c\,M = c\,(a\,M + b) = -M + c\,b$, meaning $\forall k\ M_{ck} = M_{c\,b-k}$. □

In musical terms, this means that among the augmentations of any autosimilar melody, the retrogradation of the initial melody can always be found.

3.3.3 *Inverse-retrograde symmetry.* The last situation is about melodies whose inverse IS the retrograde (like in Johnson's conjecture).

For instance, with $f : x \mapsto 3x + 1 \mod 26$ [no fixed points]:

Figure 15. Autosimilar melody with inverse-retrograde symmetry

It can be seen, and even better, heard, that $\mathcal{O}_0$ and $\mathcal{O}_8$ (resp. $\mathcal{O}_2$ and $\mathcal{O}_3$) are retrogrades one of another. This allows a pretty rendition of the melody, setting opposite **notes** for symmetric orbits: then the retrograde of the melody will be its inversion, as seen on figure 15 (the symmetry axis for pitches is around F). It is a little difficult, in fact, to find an example of an autosimilar (primitive) structure **without** such a symmetry (this is related to Tom's conjecture). For one thing, if f is an homothety (recall this happens whenever $a - 1 \mid b$ in $\mathbb{Z}_n$, for instance when $b = 0$), then $x \mapsto -x$ permutes the orbits, as all other homotheties do. Also if some power of a is equal to $c - 1$, we get directly a palindrom.

Still an autosimilar melody built from $4x + 1 \mod 21$ does the trick, as its orbits (0, 1, 5), (2, 9, 16), (3, 11, 13), (4, 6, 17), (7, 8, 12), (10, 18, 20), (14, 15, 19) exhibit no inversional symmetry whatsoever. There is a condition ensuring that such retrogradation symmetries between orbits exist:

Theorem 3.10 *Assume $a - 1$ divides $2b, 2b = c(a - 1)$, then all orbits are permuted by the symmetry $x \mapsto -c - x$, i.e.*

$$\forall x \in \mathbb{Z}/n\mathbb{Z} \qquad \mathcal{O}_{-x-c} = -c - \mathcal{O}_x$$

(some orbits may be self-invariant under this symmetry)

This condition is sufficient but not necessary. It entails entails from a generalization of the homotheties case:

Lemma 3.11 *If $f, g \in \mathrm{Aff}_n$ commute, then g permutes the orbits of f (generally: any element of the commutator of a subgroup permutes the orbits of the subgroup).*

Proof $g(\mathcal{O}_x) = \{g(f^k(x))\} = \{f^k(g(x))\} = \mathcal{O}_{g(x)}$. □

It can then easily be checked that $g = I_c : x \mapsto c - x$ will commute with $f : x \mapsto a\,x + b$ on the condition that $2b = c(a-1) \mod n$.

These examples open a new alley for future research, combining inner symmetries (the autosimilarity) of a melody with outer symmetries (e.g. inversion), using some structural features of the space of musical events.

4 Autosimilarity and tilings

This is about some mosaics, or tilings, which are deduced from some autosimilar melodies.

4.1 *Tesselations with autosimilarity*

The problem of tesselating $\mathbb{Z}_n$ is an ancient and difficult one ([1, 3, 6]). It can be asked whether an autosimilar melody tesselates $\mathbb{Z}_n$, meaning that all orbits are translates of one another.

Example 4.1 *All maps of form $x \mapsto x + b$ will give some tesselation (with only one orbit when $b \in \mathbb{Z}_n^*$).*

A less trivial case: $7x + 7 \mod 12$ has two orbits in $\mathbb{Z}_{12}$, (0 3 4 7 8 11) and (1 2 5 6 9 10), which make up a tiling.

It is rather difficult to meet the tight requirements for an autosimilar melody to be a tiling: the orbits must be, not only the same size, but also the same shape. Also the last example shows why tiles will often be periodic themselves (i.e. all invariant under some same translation): if some power of the map f is a translation $x \mapsto x + \tau$, then all orbits must be $\tau-$periodic. We have only a very partial result:

Theorem 4.2 *A (primitive) melody autosimilar with ratio $a \neq 1$ gives a tiling of $\mathbb{Z}_n$ by translations of a 2-tile when $n = 4k, a = 1 + 2k, b = \pm k$ with k odd.*

Proof Consider the orbits of 0 and 1: $\mathcal{O}_0 = (0, b), \mathcal{O}_1 = (1, a + b)$. If we have a tiling then $\mathcal{O}_0 + \tau = \mathcal{O}_1$ for some τ. This τ cannot be 1 lest $a = 1$; hence $\tau = 1-b = a+b$. We get $a = 1-2b$. Now notice $\mathcal{O}_2 = (2, 2a+b)$. If it is a translate of $\mathcal{O}_0$ then it is either by 2 or by $2 - b$. In the latter case, $0 + (2 - b) = a + b$ which leads to a contradiction. So it is $2a + b = b + 2$, hence $2a = 2$ which admits only the solution $a = 1 + n/2$ as $a = 1$ is forbidden. Notice that n must be a multiple of 4. Moreover k must be odd or else f will not be of order 2, as we see in the reverse:

Conversely, if $n = 4k, a = 2k + 1, b = \pm k$ then

$$f^2(x) = a(ax + b) + b = (4k^2 + 4k + 1)x \pm k(2k + 2) = x \qquad \text{when } k \text{ is odd and hence } 2k + 2 = 4m$$

so $o(f) = 2$, with $f(x) - x = 2kx \pm k$ being allowed only two (opposite) values.

Going from one orbit to the next is like braiding a girdle, as each is reversed from the other while translated. □

Remark 1 It is easier to build tiles with **unions** of orbits of an affine map but we have no general result. Indeed the question of tiling/autosimilar arose when Tom Johnson found $(0, 1, 3, 7, 9)$ which tiles $\mathbb{Z}_{20}$ by translation and also with ratio 3.

Remark 2 It is still an open question whether it is possible to find non periodic tiles of length > 2 (Vuza Canons) in this fashion. A necessary condition is the absence of translations in the symmetry group, and notably $o(f) = o(a)$.

4.2 *Tilings with augmentations*

In the simplest case n prime, $b = 0$, we get tilings of $\mathbb{Z}_n^*$ with augmentations as any orbit is an augmentation of $\mathcal{O}_1$. In a manner of speaking, this works for any autosimilar melody, but with overlapping, so it is not a *bona fide* tiling: for instance with $f : x \mapsto 3\,x \mod 8$ the augmentations of (1 3) are (0 0), (2 6), (4 12=4) and (5 15=7), $\mathcal{O}_4$ overlaps with itself when played as (4 12).

There is a better way to obtain a whole family of tilings with particular affine maps. **But the tiles will no longer be the orbits**: on the contrary they will be sets **transverse** to them.

LEMMA 4.3 *Any autosimilar melody whose orbits share* ***the same length*** *enables to build tilings with augmentation.*

Proof Consider any set transverse with the orbits, i.e. X containing one point and only one from each orbit. Then $X, f(X), \dots f^{r-1}(X)$ partition $\mathbb{Z}_n$ i.e. X tiles with augmentations $a\,X + b$ a.s.o. □

Example 4.4 *All the orbits of* $f : x \mapsto 13\,x + 3 \mod 20$ *have length 4: (0 2 3 9), (1 6 11 16), (4 15 17 18), (5 7 8 14) and (10 12 13 19). Take for instance the first elements: X =(0 1 4 5 10). Applying f yields all following members of each orbit:* $f(X) =$ *(3 16 15 8 13). Iteration of the process gives a mosaic, where all motives are images of the preceding one by the map* f*. Notice that one can choose any starting element in each orbit.*

Now we would like to know when this can happen. We have no simple arithmetic characterization, but a few interesting conditions. Notice that $a-1$ is not allowed to divide b (in $\mathbb{Z}_n$), which excludes $a-1$ being coprime with n in $\mathbb{Z}$, or $b = 0$, or else we have fixed points.

LEMMA 4.5 *All orbits have the same length whenever the* ***smallest*** *orbit has length (some multiple of)* $o(a)$*.*

Proof Let x_0 have the smallest orbit: $((a-1)x_0 + b)(1 + a + \dots a^{m-1}) = 0$ and

$$\forall x \; ((a-1)x + b)(1 + a + \dots a^{r-1}) = 0 \Rightarrow r \geq m$$

All orbits have same length, i.e. m, $\iff$

$$\forall x \; ((a-1)x+b)(1+a+\dots a^{m-1}) - ((a-1)x_0+b)(1+a+\dots a^{m-1}) = ((a-1)(x-x_0)+b)(1+a+\dots a^{m-1}) = 0$$

As $x(a-1)(1 + a + \dots a^{m-1}) = 0$ for m is a multiple of $o(a)$ and $((a-1)x_0 + b)(1 + a + \dots a^{m-1}) = 0$ by assumption, this works.

Other implication: if all orbits share the same length, we know it is the order of f, which is a multiple of $o(a)$. □

It boils down to a cross product between the factors of arithmetic sequences with ratio $a - 1$ and of partial sums of the geometric sequence with ratio a:

THEOREM 4.6 *All orbits will have the same length* $\iff \forall x$

$$\Big(((a-1)x + b) \times (1 + a + \dots a^{m-1}) = 0 \quad \Rightarrow \quad a^m = 1\Big) \tag{5}$$

that is to say one cannot have $((a-1)x + b) \times (1 + a + \dots a^{m-1}) = 0$ ***without*** $o(a)$ *dividing* m*.*

Proof Assume all orbits have the same length, i.e. that length is a multiple of $o(a)$. The length of $\mathcal{O}_x$ is the smallest r such that

$$a^r\,x + b \times (1 + a + \dots a^{r-1}) = x \iff ((a-1)x + b) \times (1 + a + \dots a^{r-1}) = 0$$

By assumption this is a multiple of $o(a)$. Hence $a^r = 1$. Notice this implies that $b \times (1 + a + \dots a^{r-1}) = 0$ too.

Now for any $m > 0$ with

$$((a-1)x+b) \times (1+a+\dots a^{m-1}) = 0$$

this means $f^m(x) = x$ and m is a period of f on $\mathcal{O}_x$. As seen previously this means that m is a multiple of r, length of $\mathcal{O}_x$: again $a^m = 1$.

In the other direction: assume that $((a-1)x+b) \times (1+a+\dots a^{m-1}) = 0$ implies that $a^m = 1$, i.e. $o(a) \mid m$. This means exactly that all orbit lengths are multiples of $o(a)$, by the lemma above, it proves that all orbits are the same length. □

Example 4.7 *For instance, for* $13\,x+3 \mod 20$*, the sequence of the* $1+a+\dots a^{m-1}$ *takes values 1, 14, 3, 0 cyclically. The order of* $a = 13$ *is 4* $\mod 20$*. If* $b = 3$*, one computes* $b + x(a-1) = 3, 15, 7, 19, 11$*, neither of which gives 0 when multiplied by 1, 14 or 3. So the condition is verified, as it is for* $b = 7, 11, 15$ *or* 19 *or indeed any odd* b*.*

But for (say) $b = 2$*, it does not work (*$|\mathcal{O}_4| = 2$*). The sequence* $(12\,x+2)_{x=0\dots4}$ *contains 10 (for* $x = 4$*), and this enables* $((a-1)x+b) \times (1+a+\dots a^{m-1}) = 0$ *for* $m = 2 < 4$*, i.e. an orbit of length 2 instead of 4.*

There is little hope of simplifying this condition: one has to look into the sequence $1+a+\dots a^{m-1}$ (m varies from 1 to some divisor of $o(f)$) for factors c common with n, and look for arithmetic sequences $(b + x(a-1))$ (x from 0 to $n/\gcd(n, a-1)$) that do NOT provide the missing factors n/c. In the last example, the only possible case was with $m = 2, 1+a = 14$, common factor 2: one had to find arithmetic sequences with ratio 12 with no term a multiple of $20/2 = 10$.

Remark 3 When equal length orbits are longer than $o(a)$, it means that $o(f) > o(a)$. In that case, all orbits will be periodic, as they will be invariant under $f^{o(a)}$ which is a translation (not identity by assumption). For instance with

$$f : x \mapsto 3\,x+1 \mod 8 \qquad \mathcal{O}_0 = (0,1,4,5) \quad \mathcal{O}_2 = (2,3,6,7) \qquad f^2(x) = x+4$$

But such inner symmetries cannot happen when $o(a) = o(f)$, as we have seen with $x \mapsto 13\,x+3 \mod 20$.

Remark 4 Say $a = n-1$ for some even n. $f : x \mapsto 2k+1-x \mod n$ is a map with all orbits of length 2. Omitting those 'trivial' solutions, the first values of (n,a) giving such tilings are (with adequate values for b)

$$(8,3) \quad (8,5) \quad (12,5) \quad (12,7) \quad (16,2k+1) \quad (18,7) \quad (18,13) \quad (20,9) \quad (20,11) \quad (20,13)\dots$$

Some other families of solutions could be devised likewise, e.g. when $n = 4k$, $a = 2k\pm1$, $f : x \mapsto (2k-1)x+1$ has orbits of length four. But musically (following Tom Johnson's advice) it is better to keep to small values of a, so we won't pursue this line.

5 Approximate autosimilarity

In a way, autosimilarity in a (periodic) melody is a special form of redundancy: as we have seen by now, autosimilarity is an aural illusion, where identical notes are identified though lying in fact in different positions in the original melody and in its augmentation. It is a legitimate question to ask for approximate autosimilarity: what if some melody is autosimilar apart for a few notes ? With which ratio ? It turns out this can be investigated with a simple algorithm, and also that such relaxed autosimilarity appears surprisingly often in the corpus of classical music.

5.1 *Algorithm*

Let M be some melody with period n. We define the periodic augmentations of M as the $a \times M + b$ in symbolic notation, meaning the sequences $(M_{a\,k+b \mod n})_{k\in\mathbb{N}}$. As seen above, M is autosimilar iff $a \times M + b = M$ for some a, b. We can compute a correlation coefficient between all $a \times M + b$ and M itself by checking the proportion of notes that are the same – technically it is a comparison of circular lists, or circlists, between M and $a\,M$. Checking for maximum on b, then a, allows to find the best candidate for autosimilarity. Here is an implementation in Mathematica^{TM}, with the trick for comparison that `Union` is applied to pairs $M_k, M_{a\,k+b}$ in order to count for singletons i.e. the number of coincidences:

```
correl[melo_, a_]:= Module[{n=Length[melo], meloBis},
(* local variables *)
 meloBis = Table[melo[[Mod[a*(k+decalage)-1 , n]+1]], {k,n},{decalage, n}];
 (* this is the alternate melody a*M+ decalage *)
Max[(Count[Length/@ (Union /@ Transpose[{melo, #}]), 1])& /@ meloBis]/n]
```

For instance, trying this function on a perturbed Alberti bass:

```
Table[correl[{C, G, E, A, C, G, E, G},k], { k, {3,5,7}}]
```

yields the correlation coefficients $\left\{\frac{3}{4}, 1, \frac{3}{4}\right\}$. So though 3 is no longer a ratio for autosimilarity, 5 still is.

5.2 *Example*

It is fairly obvious that no autosimilarity will be found when all notes are different – the melody cannot be broken down into orbits in that case. But with repeated rythmic motives involving repeated notes, autosimilarity may well be found. The first melodic sentence of Beethoven's fifth exhibits very good autosimilarity when cut down to 12 notes: G, G, G, E♭, A♭, A♭, A♭, G, E♭, E♭, E♭, C has a correlation coefficient of 5/6 for $x \mapsto 7\,x + 6$. Musically this means only two notes are not identical in the augmented version. These alien notes have been written down as a chord together with the 'expected' note on the score (notice though the octave identification for E♭'s).

Figure 16. A famous almost autosimilar melody

This algorithm should hopefully be run with good results on a lager corpus of classical or Jazz music, as it did strike gold on the first try. It is also part of the routines looking for 'interesting melodies' in the OMax software for improvisation.

6 About general affine maps (not one to one)

6.1 *Definition*

The condition that the ratio a be invertible modulo n may appear as a little artificial, a convenience in order to allow the mathematical tools to come in. Actually the initial definition can work well with any ratio (not zero), as for instance melody

$$\mathbf{D}, G, F, \mathbf{G}, D, G, \mathbf{F}, G, D, \mathbf{G}, F, G, \mathbf{D}, G, F, \mathbf{G}, D, G, \mathbf{F}, G, D, \mathbf{G}, F, G \ldots$$

with period 24 is certainly autosimilar with ratio 3, though 3 is not coprime with 24. Of course it will be argued that here, the smallest period (4) IS coprime with 3. So we have to consider what happens when iterating an affine map that is not bijective.

6.2 *Universal property*

There is a very satisfying theorem, establishing autosimilar melodies as universal objects.

THEOREM 6.1 *The iteration of any affine map f modulo n (not one to one) eventually reduces to iterating an affine* ***transformation*** *on some subset of $\mathbb{Z}_n$. Musically this means that one hears an autosimilar melody after several augmentations of any periodic melody. Mathematically, it means that the submelody $\widetilde{M} = f^p(M) = \left(M_{f^p(k)}\right)_{k\in\mathbb{Z}}$ is autosimilar by some power of f: $f^q[\widetilde{M}] = \widetilde{M}$ for some p, q.*

Example 6.2 *Consider this seemingly random sequence of 36 notes as a periodic melody:*

$D, C, G, G, B, F, A, B, F, H, B, E, B, B, G, H, H, C, E, C, C, E, E, E, H, H, E, A, C, E, E, E, D, F, H, E, (D, C, G \ldots)$

The two first iterations of map $x \mapsto 3x - 1 \mod 36$, that is to say picking one note out of three starting with the second, yield

$$C, B, B, B, B, H, C, E, H, C, E, H, C, B, B, B, B, H, C, E, H, C, E, H, C, B, B, B, B,$$
$$B, B, E, E, B, B, E, E, B, B, E, E, B, B, E, E, B, B, E, E, B, B, E, E, B, B, E, E, \ldots$$

the last of which is periodic and autosimilar: further iterations of the same transform will yield the same melody. We notice that several notes have disappeared, and that the ultimate period is smaller than 36.

Proof The set (algebraically, a monoid) of all affine maps modulo n is finite. Powers of f thus will only take a limited number of different values. So there must exist two different exponents $p, p + q$ with $f^p = f^{p+q}$. Now for any $r > p$,

$$f^{r+q} = f^{(p+q)+(r-p)} = f^{p+q} \circ f^{r-p} = f^p \circ f^{r-p} = f^{p+(r-p)} = f^r$$

We have shown a classical result: the sequence of powers of f is ultimately periodic. So is for any $x \in \mathbb{Z}_n$, the sequence $f^k(x)$. This means that after p iterations of f, any further iteration of f^q will preserve the sequence. □

Now the algorithm that enables to construct such an **ultimately autosimilar melody** is straightforward:

Definition 6.3 *We generalise the definition of orbits to the* ***attractor*** *of* x*:* $\mathcal{A}_x = \{f^k(x) \mid k \geq p)$*. It is the part of the sequence* $f^k(x)$ *that loops (beware ! usually* $x \notin \mathcal{A}_x \ldots$*).*

Now it only remains to ensure that, as in the preceding theorems about building up autosimilar melodies, all notes with indexes in the same A_x are identical. In the above example, we have two attractors, $A = (5, 14)$ and $B = (23, 32)$. The initially completely random melody M was modified is setting $M_{14} = M_5 = B, M_{23} = M_{32} = E$. All other notes are irrelevant. Musical applications could involve extracting a simple, autosimilar beat, from a complex melody. Another nice application is to arrange the initial melody in order to support several extractions of ultimately autosimilar melodies. For instance,

$$E, H, B, G, C, G, H, G, E, A, H, H, C, C, B, G, F, G, C, G, H, G, G, H, A, F, G, G, E, H, H, G, F, G, F, B \ldots$$

gives **two** autosimilar melodies when augmented by 2 or by 3 by applying the same trick as above both to the attractors of $x \mapsto 3\,x + 1$ and those of $x \mapsto 2\,x$. It is plainly visible on the 'score' below that two (simple) autosimilar melodies emerge (the initial melody is the middle voice).

Figure 17. Two attractors for one melody

6.3 *A Fitting ending*

We will round up this last theorem with a more detailed explanation in the simpler case of homotheties, which links this result with the abstract Fitting Lemma already connected with several musicological situations (Anatol Vieru's iteration of the difference operator, [4])[1]. A connection to the general case is that the ultimate period of $f \in \mathrm{Aff}_n : x \mapsto a\,x + b$ is a multiple of the ultimate period of its linear part $\vec{f} : x \mapsto a\,x$.

Let us consider this map $x \mapsto a\,x \mod n$. First we will assume for simplicity's sake that $\gcd(a, n) = p$ is a prime factor. This means $n = p^m q$ where q is coprime with p. Now $x \mapsto a\,x$ maps $\mathbb{Z}_n$ into the cyclic subgroup of index p, namely $p\,\mathbb{Z}_n$, isomorphic with $\mathbb{Z}_{p^{m-1}q}$. After m iterations we are working in $p^m\,\mathbb{Z}_n$, cyclic subgroup of $\mathbb{Z}_n$ isomorphic with $\mathbb{Z}_q$. There $x \mapsto a\,x$ is one-to-one, at long last.

Proposition 6.4 *Attractors* $A_x = \{a^k x, k \geq m\}$ *are the orbits of* $x \mapsto a\,x$ *operating on* $\mathbb{Z}/q\mathbb{Z}$*, identified to subgroup* $p^m(\mathbb{Z}/n\mathbb{Z})$*.*

At this juncture, everything is like in section I: f cycles around the $\mathcal{A}_x$, generating an autosimilar melody. In a more general setting, this is a case of the Fitting Lemma, a very abstract result on decomposition of modules in commutative algebra:

[1] It is worth noticing that orbits, for homotheties, or their difference sets, for general affine maps, are exactly the eigenvectors of Vieru's difference operator acting on subsets of $\mathbb{Z}_n$: $\Delta(x, a\,x, a^2\,x \ldots) = (a-1)(x, a\,x, a^2\,x \ldots)$.

THEOREM 6.5 *Let $p_1, \dots p_r$ be the prime factors belonging to both a and n:*

$$a = p_1^{m_1} \dots p_k^{m_k} \ (k \geq r) \qquad n = p_1^{n_1} \dots p_k^{n_k} \times Q = P \times Q, \ \textit{with} \ \gcd(P, Q) = 1$$

then the sequence $(a^k\, x)_{k \in \mathbb{N}}$ is ultimately periodic, from at least the rank r verifying $(n/q) \mid a^r$, i.e. the smallest integer exceeding all ratios n_i/m_i.

The periodic parts of this sequence, i.e. the attractors $\mathcal{A}_x = \{a^k\, x, k > r\}$, partition the sub-group $\dfrac{n}{q}\mathbb{Z}_n$ of $\mathbb{Z}_n$, isomorphic with $\mathbb{Z}_q$.

Proof We use the Chinese Remainder Theorem: the ring $\mathbb{Z}_n$ is isomorphic with the ring product $\mathbb{Z}_P \times \mathbb{Z}_Q$ (meaning essentially that any residue class modulo n is well and truly determined by its residues modulo P and modulo Q). Thus any (affine) map in $\mathbb{Z}_n$ can be decomposed into two (affine) components on $\mathbb{Z}_P$ and $\mathbb{Z}_Q$: if $x \in \mathbb{Z}_n$ corresponds to $(y, z) \in \mathbb{Z}_P \times \mathbb{Z}_Q$ then $f(x)$ corresponds with $(\widehat{f}(y), \widetilde{f}(z))$ where $\widehat{f}(y) = f(x) \mod P, \widetilde{f}(z) = f(x) \mod Q$.

Now for map $f : x \mapsto a\, x \in \mathbb{Z}_n$: as $a^r = 0 \mod P$ we have $\widehat{f}^r = 0$ (the null map), i.e. $\widehat{f}$ is nilpotent; conversely, as a is coprime with Q, the other component $\widetilde{f}$ is one to one. After $k \geq r$ iterations, f^k reduces to $(\widehat{f}^k, \widetilde{f}^k) = (0, \widetilde{f}^k)$ and we are back to section 1. □

Remark 1 The result of r iterations of f on the original melody is not necessarily invariant under f itself, but as proved above, it is always invariant under f^τ where τ is the order of $\widetilde{f}$. In other words, it has at most τ 'alternate melodies' (under iteration of f). Though in theory one might stumble on the case $f^\tau = id$ (for instance if the original melody has n distinct notes), in practice one frequently does find some non trivial attractor. In this sense, autosimilar melodies are exactly the attractors of any affine map operating on any initial (periodic) melody, thus reaching the elated status of universal object.

Acknowledgements

First and foremost I must thank and congratulate composer Jom Johnson for his pioneering work on the subject and the wonderful music he managed to make out of this basically simple idea. Also I am grateful to Gerard Assayag who introduced me to the notion over a glass, and later put the question of detecting approximate autosimilarity. Carlos Agon implemented it in *OpenMusic*^{TM} and is therefore responsible for several nice illustrations in this paper, while Moreno Andreatta helped clarify a number of less than obvious points.

References

[1] Amiot, E., *Why Rhythmic Canons are Interesting*, in: E. Lluis-Puebla, G. Mazzola et T. Noll (eds.), *Perspectives of Mathematical and Computer-Aided Music Theory, EpOs*, 190–209, Universität Osnabrück, 2004.

[2] Amiot, E., Agon, C., Andreatta, M., *Implementation of autosimilar melodies in OpenMusic.*

[3] Andreatta, M., *Méthodes algébriques en musique et musicologie du XXe sicle : aspects thé oriques, analytiques et compositionnels* , (2003) Ph.D. dissertation, EHESS.

[4] Andreatta, M., Agon, C., Vuza, D.T., *Analyse et implmentation de certaines techniques compositionnelles chez Anatol Vieru*, in *Actes des Journes dInformatique Musicale*, Marseille, (2002), pp. 167-176

[5] Feldman, D., *Self-Similar melodies*, in *Leonardo Music Journal*, (1998) 8:80-84.

[6] Fripertinger, H., *Tiling Problems in Music Theory* in in: E. Lluis-Puebla, G. Mazzola et T. Noll (eds.), *Perspectives of Mathematical and Computer-Aided Music Theory, EpOs*, 153–168, Universität Osnabrück, 2004.

[7] Johnson, T., *Self-Similar melodies*, (1996), Two-Eighteen Press, NY.

[8] Rahn, J., *Basic Atonal Theory*, (1980), Longman, New York.

[9] Morris, R., *Composition With Pitch Classes: A Theory of Compositional Design*, (1987), New Haven, Yale University Press.

[10] Mazzola, G., et alli, *Topos of Music* (2004), Birkhaüser.

Figure 18. Loops for Orchestra