A Frobenius group is a transitive permutation group F on a finite set X such that no member of $F^{\#}$ fixes more than one point of X and some member of $F^{\#}$ fixes at least one point of X.

THEOREM (Frobenius). If $H \subset F$ (F finite), $H^a \cap H = 1$ for all $a \in F \backslash H$, and K is the set of elements of F not in any conjugate of $H^{\#}$, then K is a normal subgroup of F.

THEOREM (Zassenhaus). For each odd prime p, the Sylow p-subgroups of F_x are cyclic, and the Sylow 2-subgroups are either cyclic or quaternion. If F_x is not solvable, then it has exactly one nonabelian composition factor, namely A_5.

THEOREM (Scott). If F is a finite transitive permutation group such that if $x \in F^{\#}$, then $\mathrm{Ch}(x) = 0$ or 1, then the set $K = \{x \in F \mid \mathrm{Ch}(x) = 0 \text{ or } x = 1\}$ is a regular, normal subgroup of F.

Structure of the Frobenius group of order 20:

A presentation of the group is $F = \langle c, f \mid c^5 = f^4 = 1, cf = fc^2\rangle$.

$c = (1,2,3,4,5)$
$f = (1,2,4,3)$

F is a one-dimensional affine group $\mathrm{AGL}_1(G)$ over some (commutative) field G.

F is sharply 2-transitive.

$F \cong \mathrm{Aut}(F) = \mathrm{Inn}(F)$.

F contains a fixed-point free subgroup $K = \langle c \mid c^5 = 1\rangle \cong C_5$, called the *Frobenius kernel*. (In this context, it consists only of translations.) When H has even order, K is abelian. It is regular, hence $|K| = |X|$, and normal in (a finite) F. Each non-trivial element in K has the same order (5), by F's being 2-transitive. It is also the centralizer $C_F(c)$ of any $c \in K$. K is a nilpotent group. $\mathrm{Aut}(K) \cong F/K \cong C_4$.

F contain a proper nontrivial subgroup $H = \langle f \mid f^4 = 1\rangle \cong C_4$, which fixes a point $x \in X$. H is called the *Frobenius complement*. It (the conjugacy class of stabilizers) is a trivial intersection set (TI-set) in F. It is abelian. It is also the centralizer $C_F(f)$ of any $f \in H$. H is its own normalizer: $H = N_F(H)$. It is determined up to conjugacy in F. $H = F_x$ acts regularly on each of its orbits on $X \backslash \{x\}$. H has exactly one element of order 2 (when H has even order). When X is finite, this implies that $|F_x|$ divides $|X|$-1. $|\mathrm{Aut}(H)| = 2$.

F is faithfully represented as a Frobenius group by right multiplication on the coset space F/H.

Distinct elements of K lie in distinct right F_x-cosets (for some, or all $x \in X$).

The right regular representation of F yields a simply-transitive group F' that acts on F.

For all $x, y \in X$, there is at most one element $k \in K$ such that $x^k = y$.

F is a semidirect product HK of K by H (by K's being normal, and H's being a complement to K).

H acts semiregularly on K; that is, $C_H(k) = 1$ for each $k \in K^\#$, or equivalently $C_K(h) = 1$ for each $h \in H^\#$.

F contains a nonabelian subgroup $D = \langle f^2, g^2 \mid f^4 = g^4 = (f^2g^2)^5 = 1\rangle \cong D_{10}$. D is normal in F. $|F/D| = 2$. $\mathrm{Aut}(D) \cong F$, and $\mathrm{Inn}(D) \cong D$.

Conjugacy classes:
$[1]$, $[c] = [c^2] = [c^3] = [c^4]$, $[f] = [g] = [h] = [k] = [m]$, $[f^2] = [g^2] = [h^2] = [k^2] = [m^2]$, $[f^3] = [g^3] = [h^3] = [k^3] = [m^3]$