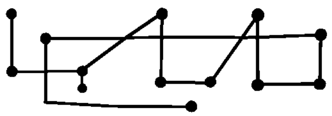
SELF-SIMILAR PITCH STRUCTURES, THEIR DUALS, AND RHYTHMIC ANALOGUES



NORMAN CAREY
AND
DAVID CLAMPITT

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the Palm of your Hand
And Eternity in an hour

William Blake, “Auguries of Innocence”

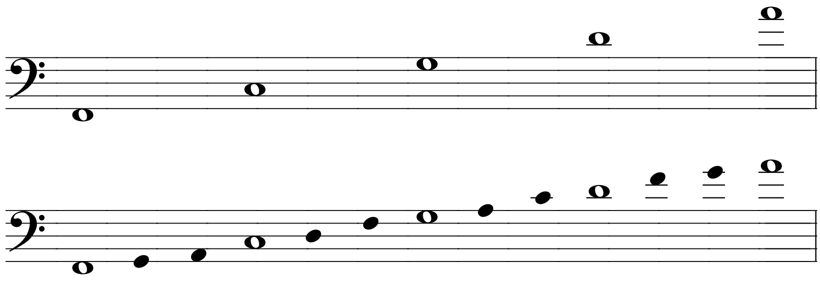
A SELF-SIMILAR STRUCTURE is one that exhibits parallel construction at different levels of scale. Notions of self-similarity have often been invoked in organicist explanations of the evolution and unity of musical compositions. At around the same time Blake wrote the quatrain that serves as an epigraph for our paper, Heinrich Christoph Koch was offering his conception of musical form as the expansion and elaboration of what he considered the basic musical element, the phrase. The most profound attempt in this direction is the mature theory of Heinrich

Schenker: aspects of self-similarity are evident in his concepts of *Schichten* and *verborgene Wiederholung*. A few years back, Charles Wuorinen and Benoit Mandelbrot presented a joint lecture-concert at the Guggenheim Museum on fractals and music, perhaps the most notable of many such experiments. It is the appeal of fractals and chaos theory that has revived interest in self-similarity in scientific circles, where the notion had long been in disrepute.

We will show how the diatonic scale exhibits a kind of self-similarity, and prove that this self-similarity property characterizes a class of scales sharing some essential features of the diatonic. This proof in turn suggests the notion of a self-similar dual. We also present a compositional idea for realizing analogues of self-similar pitch structures in the rhythmic domain. The ultimate, unrealizable model is an infinite sequence, that potentially exhibits two kinds of self-similarity, in what we may call horizontal and vertical dimensions.

By *interval of periodicity* we mean an interval whose two boundary pitches are functionally equivalent. Normally, the octave is the interval of periodicity. By *scale* we refer to a set of pitches ordered according to ascending frequencies (pitch height) bounded by the interval of periodicity. A *step interval* is an interval whose two boundary pitches are adjacent pitches of a scale.

The self-similarity found in the diatonic and in several other well-known scale systems proves to be associated with the property called *well-formedness*. Well-formed scales were defined in Carey and Clampitt 1989.[1] A generated scale is well-formed if its generator always spans the same number of step intervals. As Example 1 shows, the pentatonic scale is generated by the perfect fifth, and is well-formed because all four of its generating perfect fifths span three steps.



The pentatonic scale, generated by the perfect fifth, is well-formed since all perfect fifths span the same number of steps (3).

EXAMPLE 1

Any equal-interval scale, such as the ordinary twelve-note chromatic, trivially satisfies the definition of well-formedness, and is referred to as *degenerate well-formed*. In the usual mod 12 universe, the most interesting well-formed sets are those generated by ic 5, especially the pentatonic and diatonic, but there are in all twenty-one nondegenerate well-formed sets within the usual twelve-note universe. These are listed in Example 2.

no.	Forte number	Prime form	ic generator
1.	2–1	[01]	1
2.	2–2	[02]	2
3.	2–3	[03]	3
4.	2–4	[04]	4
5.	2–5	[05]	5
6.	3–1	[012]	1
7.	3–6	[024]	2
8.	3–9	[027]	5
9.	3–10	[036]	3
10.	4–1	[0123]	1
11.	4–21	[0246]	2
12.	5–1	[01234]	1
13.	5–33	[02468]	2
14.	5–35	[02479]	5
15.	6–1	[012345]	1
16.	7–1	[0123456]	1
17.	7–35	[013568a]	5
18.	8–1	[01234567]	1
19.	9–1	[012345678]	1
20.	10–1	[0123456789]	1
21.	11–1	[0123456789a]	1

EXAMPLE 2: THE 21 NONDEGENERATE WELL-FORMED SETS IN THE TWELVE-NOTE UNIVERSE

Given an interval in the context of some particular scale, the number of step intervals that the interval spans is its *generic* measure. In other words, generic measure is the number of step intervals enclosed by the interval. In all nondegenerate well-formed scales, each nonzero generic interval class contains exactly two specific varieties. Clough and Myerson 1985 introduced the term *Myhill's Property*, or *MP*, to designate this property. Example 3 presents a tabulation of the constituent steps of diatonic intervals. The two specific intervals which span the same number of steps appear in the same row. Such a decomposition of the diatonic intervals is found in Hucbald, albeit in garbled form, and in later treatises such as the *Contrapunctus* of Prosdocimo de' Beldomandi.[2]

span	smaller	no. of m2s	no. of M2s	larger	no. of m2s	no. of M2s
1	m2	1	0	M2	0	1
2	m3	1	1	M3	0	2
3	P4	1	2	A4	0	3
4	d5	2	2	P5	1	3
5	m6	2	3	M6	1	4
6	m7	2	4	M7	1	5

EXAMPLE 3: DECOMPOSITION OF DIATONIC INTERVALS INTO STEP INTERVALS

The diatonic scale is self-similar in the following respect: the distribution of semitones within any diatonic interval is approximately equal to the overall distribution of semitones within the octave, namely two in seven (see Example 4).

Span		Larger				Smaller
1	2nds	0/1	<	2/7	<	1/1
2	3rds	0/2	<	2/7	<	1/2
3	4ths	0/3	<	2/7	<	1/3
4	5ths	1/4	<	2/7	<	2/4
5	6ths	1/5	<	2/7	<	2/5
6	7ths	1/6	<	2/7	<	2/6

EXAMPLE 4: DISTRIBUTION OF HALF STEPS IN DIATONIC INTERVALS COMPARED WITH HALF STEPS PER OCTAVE

Consider two diatonic segments, for example C D E F G A and E F G A B C, spanning major and minor sixths, respectively. One of the five steps in the major sixth is a semitone, while the ratio is two in five in the case of the minor sixth. Now $1/5 < 2/7 < 2/5$, therefore $1/5$ and $2/5$ are, in fact, the closest approximations to $2/7$ with denominator 5. Example 4 shows that the same holds true for seconds, thirds, fourths, fifths, and sevenths as well.

Diatonic scale segments thus possess a synechdochic property: the part reflects the organization of the whole with a minimal, but inevitable degree of distortion. The subtle, sober variety that distinguishes Gregorian chant is, rhythmic considerations aside, a manifestation of this property of the diatonic system. The self-similarity property entails that diversity is conjoined with homogeneity, amplifying the discussion of the dialectic of *pattern matching* and *position finding* in Browne 1981. The property is also suggestive of the *maximal evenness* property defined in Clough and Douthett 1991. However, not all maximally even sets have Myhill's Property, and sets with Myhill's Property are not always maximally even.

Myhill's Property proves to be crucial for the existence of the kind of self-similarity demonstrated above. It is our intention to show that all nondegenerate well-formed scales are self-similar. We will show that a scale has Myhill's Property (*MP*) if and only if it is nondegenerate well-formed (*WF**), and that it has self-similarity (*SS*) if and only if it has Myhill's Property, or $SS \leftrightarrow MP \leftrightarrow WF^*$.

THEOREM. $SS \leftrightarrow MP \leftrightarrow WF^*$

We require some preliminary definitions and formalisms. Example 5 relates the symbols we will be using to the definitions, and gives examples from the diatonic scale for reference in translating into musical terms. Greek letters represent real numbers, while Latin letters are reserved for integer values.

The integral part $[\theta]$ of a real number θ is the greatest integer less than or equal to θ. The fractional part $\{\theta\}$ of a real number θ is the difference between the number itself and its integral part. That is, $\{\theta\} = \theta - [\theta]$.

Any type of scale in which the usual notion of octave equivalence is operative has a representation as a set of values between 1 and 2, $1 = f_0 < f_1 < \ldots < f_{N-1} < 2$ where the f_i are the frequencies of pitches within a representative octave, or logarithmically as $0 = s_0 < s_1 < \ldots < s_{N-1} < 1$, where $s_i = \log_2(f_i)$.

Symbol	Definition	Diatonic Example
N	cardinality of set with *MP*	7
d	span	2 = span of a third
α_d	two sizes of	Maj 3rd
β_d	intervals of span d	min 3rd
m	multiplicity of β_d	4 (no. of min 3rds)
g	multiplicity of β_1	2 (no. of min 2nds)
x	β_1 content of α_d: no. of β_1s in α_d	0 (no. of min 2nds in Maj 3rd)

EXAMPLE 5: SYMBOLS USED IN PROOF

Let S represent such a set of logarithmic values:

$$S = \{S_i | 0 = s_0 < s_1 < \ldots < s_{N-1} < 1\}.$$

We will call the ordered pairs of S the *intervals* of S. The *span* of an interval from s_i to s_j is defined to be the least non-negative value $(j-i)_{\text{mod } N}$, that is, just the number of step intervals from s_i to s_j, "wrapping around" if necessary. The *size* of an interval (s_i,s_j) is the fractional part, $\{s_j - s_i\}$. The size of an interval is thus some real number ρ, $0 \le \rho < 1$.

The two mathematical values, span and size, correspond to the musical notions of generic and specific intervals, respectively. In this setting, S is said to have Myhill's Property if for every nonzero span d intervals of S come in exactly two sizes. This is a slight but important generalization of Clough and Myerson's definition, in that we allow interval sizes to be irrational values.

The following definitions apply to sets with MP:

Let α_d and β_d designate the two interval sizes for intervals of span d. The number of distinct intervals of span d and size α_d will be called the *multiplicity* of α_d and likewise for β_d. The symbol m will designate the multiplicity of β_d. For example, in the diatonic set, if d is 2, α_2 represents the interval size "major third" and β_2 represents the interval size "minor third." Here m, the multiplicity of β_2, is 4 since there are four minor thirds. In the C-major set these are: D–F, E–G, A–C, B–D. Clearly, the total number of intervals of span d is N, so the multiplicity of α_d is $N - m$. Intervals of span 1 will be called *steps*. Let g be the multiplicity of β_1. (Note that g is a special case of m. Strictly speaking, since m varies with d, we might write m_d; then $g = m_1$, but the multiplicity of β_1 will play a special role, so we give it its own symbol.)

The property *SS* is defined exclusively on *MP* scales. Let x and x' be the number of β_1 intervals in α_d and β_d respectively. An *MP* scale is said to have *SS* if $x' = x + 1$ and

$$\frac{x}{d} < \frac{g}{N} < \frac{x'}{d}$$

for all d, $1 \le d \le N$.

PART ONE. *S* is *WF** if and only if *S* has *MP*.

Carey and Clampitt 1989 proved one half of this proposition, namely that all nondegenerate well-formed scales have *MP*. Here we prove the other half: if *S* has *MP*, then it is *WF**.

Method. By hypothesis, there are two interval sizes for intervals of span d. Since α_d and β_d exist for all $d \neq 0$, m is strictly between 0 and N. It will be sufficient to exhibit an interval of unique multiplicity: if for some span d there exists exactly one interval of size β_d, then there are $N - 1$ intervals of size α_d. Therefore, α_d may be construed as a generator of constant span d of the set *S*, so *S* is *WF**.

Proof. Recalling that g is the multiplicity of β_1, then

$$(N - g)\alpha_1 + g\beta_1 = 1. \tag{1.1}$$

The sum of the sizes of all intervals of span d equals d. This is true whether the set has MP or not. Each step is contained in d distinct intervals of span d, therefore:

$$(N - m)\alpha_d + m\beta_d = d. \tag{1.2}$$

Our next observation is that intervals of a given span and size may be uniquely decomposed into step intervals.

Suppose the contrary. Then $\alpha_d = (d - x)\alpha_1 + x\beta_1 = (d - x')\alpha_1 + x'\beta_1$, with $x \neq x'$. Then $0 = (\beta_1 - \alpha_1)(x - x')$, but since $\beta_1 \neq \alpha_1$, we must have $x = x'$, contrary to supposition. We are justified, then, in distinguishing α_d and β_d according to their step contents. Henceforth, α_d will designate the interval size which contains fewer β_1 steps, β_d the one containing more β_1 steps. Note that all arguments have been symmetric thus far with respect to α_d and β_d. It is possible to distinguish between the step intervals on the basis of size or multiplicity; it happens that in our concrete diatonic example we have arbitrarily assigned β_1 to the minor second, the step interval that is both smaller and rarer, but one should make no generalizations based upon this. It follows, given our decision to designate by α_d the specific interval of span d with fewer β_1 steps, that if $\alpha_1 > \beta_1$, then $\alpha_d > \beta_d$. It does not follow that, if α_1 has greater multiplicity than β_1, the multiplicity of α_d is greater than the multiplicity of β_d.

Next we show that there is a constant difference between the two interval sizes of a given span:[3]

$$\beta_d - \alpha_d = \beta_1 - \alpha_1 . \tag{1.3}$$

Since there must be two intervals having the same span and different sizes, we must be able to choose an interval r such that $r = (s_i, s_{(i+d) \bmod N})$ has the size α_d and $r' = (s_{(i+1) \bmod N}, s_{(i+d+1) \bmod N})$ has size β_d. Now interval r' is simply r plus one step, and minus one step: $r' = r + (s_{(i+d) \bmod N}, s_{(i+d+1) \bmod N}) - (s_i, s_{(i+1) \bmod N})$. By construction, r' contains more β_1 steps than r, so the only possibility is that $\beta_d = \alpha_d + \beta_1 - \alpha_1$, or $\beta_d - \alpha_d = \beta_1 - \alpha_1$. Therefore, if x is the β_1 content of α_d, then

$$\alpha_d = (d - x)\alpha_1 + x\beta_1$$

and (1.4)

$$\beta_d = (d - x - 1)\alpha_1 + (x + 1)\beta_1 .$$

Substituting 1.4 into 1.2,

$$\begin{aligned} d &= (N - m)((d - x)\alpha_1 + x\beta_1) + m((d - x - 1)\alpha_1 + (x + 1)\beta_1) \\ &= N(x\beta_1 + (d - x)\alpha_1) + m(\beta_1 - \alpha_1) \\ &= (\beta_1 - \alpha_1)(Nx + m) + Nd\alpha_1 . \end{aligned}$$

So $\dfrac{d(1 - N\alpha_1)}{Nx + m} = \beta_1 - \alpha_1$ and from 1.1, $\dfrac{1 - N\alpha_1}{g} = \beta_1 - \alpha_1$.

Thus,

$$\frac{d(1 - N\alpha_1)}{Nx + m} = \frac{1 - N\alpha_1}{g} .$$

We would like to simplify this equation by factoring out the value $(1 - N\alpha_1)$. This value must be nonzero, since otherwise all N of the step intervals would be of size α_1, contradicting the assumption that the set has *MP*. So,

$$\frac{d}{Nx+m} = \frac{1}{g}, \text{ or:}$$

$$dg = Nx + m. \qquad (1.5)$$

We assert that g is a *unit* modulo N, that is, $(g, N) = 1$. Suppose $(g, N) = p > 1$. Then $1 < \frac{N}{p} < N$. Let $d = \frac{N}{p}$.

Then, since $dg = Nx + m$, $\frac{N}{p}g = Nx + m$, so $\frac{g}{p} = x + \frac{m}{N}$. But $\frac{g}{p}$ is an integer, as is x, contradicting $0 < m < N$. Then g is a unit mod N. From this information, together with 1.5 we have therefore:

$$dg \equiv m_{\text{mod } N}, \qquad (1.6)$$

and setting $d = g^{-1}{}_{\text{mod } N}$, $m = 1$. That is, when $d = \frac{1}{g}\text{mod } N$, β_d has multiplicity 1, α_d multiplicity $N - 1$. It follows that S is well-formed. Thus $MP \leftrightarrow WF^*$.

In the diatonic, $g = 2$, $N = 7$, and the multiplicative inverse mod 7 of 2 is 4, which is indeed the span of the unique diminished fifth, and of the generating perfect fifths.

PART TWO. *S* has *SS* if and only if *S* has *MP*.

Method. Since *SS* scales have *MP* by definition, we need only prove that *MP* implies *SS*.

Proof. Returning to line 1.5 above, we have: $dg = Nx + m$, where x is the β_1 content of α_d.

Dividing both sides of equation 1.5 by dN, we have:

$$\frac{g}{N} = \frac{x}{d} + \frac{m}{Nd} \qquad (2.1)$$

Clearly, $\frac{m}{Nd}$ is a positive number less than $\frac{1}{d}$, because $\frac{m}{N}$ is less than 1, so we have the inequality:

$$\frac{x}{d} < \frac{g}{N} < \frac{x+1}{d}. \qquad (2.2)$$

Recall that x is the β_1 content of α_d, and so $x+1$ is the β_1 content of β_d. This fact, together with 2.2, shows that the distribution of a given step interval within an interval of any span approximates as well as possible the distribution in the whole set. Then *MP* implies *SS*. Thus we have $SS \leftrightarrow MP$.

This result, together with that of Part One, proves the theorem: $SS \leftrightarrow MP \leftrightarrow WF^*$.

* * *

There is no mathematical reason to prefer the fifth to the fourth as the generator of the diatonic scale. We might rightly suspect that there will always be two generators in a *WF** scale, whose sizes sum to an octave, and whose spans sum to *N*. The following proves the existence of this second generator and computes its span:

Since $-g_{\text{mod } N}$ (i.e., $N-g$, the multiplicity of α_1) is also a unit mod N, when $d = -g^{-1}{}_{\text{mod } N}$ (the multiplicative inverse of $N-g$), β_d has multiplicity $N-1$, α_d is unique. Thus, $\beta_{-g^{-1}{}_{\text{mod } N}}$ is also a generator of constant span.

We have, then, the following corollary:

COROLLARY. *In an MP (therefore WF*) set of N elements, the spans of the generating intervals are the multiplicative inverses mod N of the multiplicities of the step intervals.*

In the usual diatonic the multiplicities of the steps are 2 and 5. As we saw above, the multiplicative inverse of 2 mod 7 is 4, and 4 is the span of the generating interval of the perfect fifth. Similarly, 5^{-1} mod 7 is 3, and 3 is the span of the perfect fourth.

Note that by the end of the proof (and in this corollary) all of the Greek letters, and therefore all of the real numbers and their concomitant references to specific tunings, have dropped out. This makes it possible to classify nondegenerate well-formed scales based upon their characteristic abstract patterns. We use the angle-bracket notation from Morris 1987 to represent cyclic permutations of step interval patterns. For example, in any tuning of the diatonic scale, the essential pattern of step intervals is <*aabaaab*>. Moreover, any *WF** scale with step multiplicities 2 and 5 will exhibit this pattern, hence we can define an equivalence class of *WF** scales that share this pattern up to rotation. In fact, we can partition all *WF** scales into equivalence classes according to either their essential patterns or their step multiplicities, since the patterns depend entirely upon the step multiplicities.

It will be useful to let such a class be represented by a 2×2 array, displaying two pairs of interrelated integer parameters,

$$\begin{array}{|c|c|}\hline g & -g \\ \hline g^{-1} & -g^{-1} \\ \hline\end{array}_{\bmod N} .$$

The multiplicities of the step intervals appear in the top row, the spans of the generating intervals in the bottom row. Each row contains a pair of numbers which are additive inverses modulo N, just as numbers in the same column are multiplicative inverses. The diatonic belongs to the class whose values are

$$\begin{array}{|c|c|}\hline 2 & 5 \\ \hline 4 & 3 \\ \hline\end{array} ,$$

which class contains all scales whose pattern of steps is some cycle of <*aaabaab*>. The mutual relationship between steps and generators is illustrated as follows: the sum of two fifths, modulo the octave, reduces to one step, while four steps span a fifth. Similarly, five fourths reduces to one step, while three steps span a fourth.

To obtain a seven-note scale with this pattern in which there are five *larger* step intervals and two *smaller* ones only requires that the generating interval (the fifth) must have a logarithmic size strictly between $4/7$ and $3/5$ of the octave. (Expressed in cents, the fifth must be strictly between $685\ 5/7$ and 720.)

When columns are exchanged, transforming

$$\begin{array}{|c|c|}\hline g & -g \\ \hline g^{-1} & -g^{-1} \\ \hline\end{array}_{\bmod N}$$

into

$$\begin{array}{|c|c|}\hline -g & g \\ \hline -g^{-1} & g^{-1} \\ \hline\end{array}_{\bmod N} ,$$

the class represented is unchanged. On the other hand, exchanging rows establishes, in general, a different class:

$$\begin{array}{|c|c|}\hline g^{-1} & -g^{-1} \\ \hline g & -g \\ \hline \end{array}_{\bmod N} .$$

Given a class of *WF** scales, we define its *dual* to be that class in which parameters which are multiplicative inverses mod N exchange roles. That is, the dual of the class with parameters

$$\begin{array}{|c|c|}\hline g & -g \\ \hline g^{-1} & -g^{-1} \\ \hline \end{array}_{\bmod N}$$

is the class with parameters

$$\begin{array}{|c|c|}\hline g^{-1} & -g^{-1} \\ \hline g & -g \\ \hline \end{array}_{\bmod N} .$$

The 2×2 array for the class which is dual to the diatonic is

$$\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 5 \\ \hline \end{array} .$$

The pattern of steps in the dual is <*abababb*>. The step multiplicities are 4 and 3; one of the generators is a "third," spanning two steps, while the other is a "sixth," spanning five steps.

We construct a concrete *WF** scale having these parameters. Taking zero and the first six multiples of 350 cents, reduced modulo the octave, gives the set S = (0, 200, 350, 550, 700, 900, 1050). The generators, whose sizes are 350 and 850 cents, span the requisite number of steps, namely 2 and 5. The pattern <*abababb*> represents the ordering of the steps, replacing each "*a*" with an interval of 200 cents and each "*b*" with one of 150 cents.[4]

It is interesting to observe that if any rotation of the dual pattern is superimposed upon the notes of a diatonic scale, the *a*s and *b*s of the dual will partition the diatonic scale into a triad and its complement, a seventh chord. This suggests that the dual may play some useful role in the harmonic organization of tonal music.[5]

It is possible, of course, that $g = g^{-1}$ or $g = -g^{-1}$. In these cases, the pattern is its own dual. Such is the situation in the Pythagorean chromatic scale and its class,

7	5
7	5

,

or the class which contains the "black-key" pentatonic scales,

3	2
2	3

.

* * *

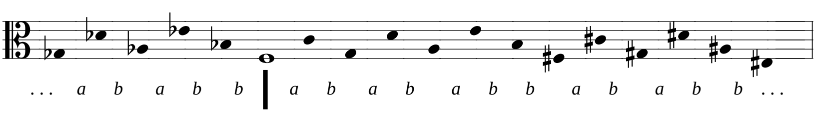
We have defined a notion of self-similarity on finite sets. In mathematical settings, the concept of self-similarity is usually applied to infinite or potentially infinite sets. As before, we represent a scale as a set of values lying between 0 and 1. Given a generating interval of size θ, $0 < \theta < 1$, a generated scale of cardinality N is represented by the set of fractional parts of non-negative integer multiples of θ: 0, $\{\theta\}$, $\{2\theta\}$, $\ldots\{(N-1)\theta\}$. For certain cardinalities N, the resulting sets are well-formed. If θ is a rational number A/B in lowest terms, it is easy to see that there are at most B distinct points generated. If θ is irrational, however, an infinite set of points may be generated, letting N range over all integers. This set is dense and uniformly distributed, according to Kronecker's Theorem (Hardy and Wright 1979). This means that nowhere is it possible to consider a "next note" using the usual order relation, and so the notion of a scale is meaningless here.

On the other hand, suppose that θ is irrational and consider the sequence of integers $[n\theta]$, that is, the integral parts of $n\theta$, as n ranges over all integers. Adjacent elements of this sequence always differ by j or $j+1$, for some integer j. Example 6 shows the sequence of integers $[n \log_2 3/2]$ as n ranges over all integers. Adjacent elements of this sequence differ by 0 or 1. We derive a *binary sequence* in the following way: if two adjacent integers differ by 0, they are replaced by an *a*, and if they differ by 1, they are replaced by a *b*.

n:	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10	11	12
$\left\lfloor n \log_2 \frac{3}{2} \right\rfloor$:	–3	–3	–2	–2	–1	0	0	1	1	2	2	3	4	4	5	5	6	7
binary sequence:	*a b a b b* | *a b a b a b b a b a b b*																	

EXAMPLE 6: BINARY SEQUENCE ASSOCIATED WITH $\log_2 \frac{3}{2}$

The *a*s and *b*s of the binary sequence are merely tokens which could admit various interpretations. A musical interpretation of the binary sequence determined by $\log_2 3/2$ is displayed in Example 7.



a up a fifth
b down a fourth

EXAMPLE 7: THE BINARY SEQUENCE FOR $\log_2 \frac{3}{2}$ DERIVED VIA TUNING PROCEDURE

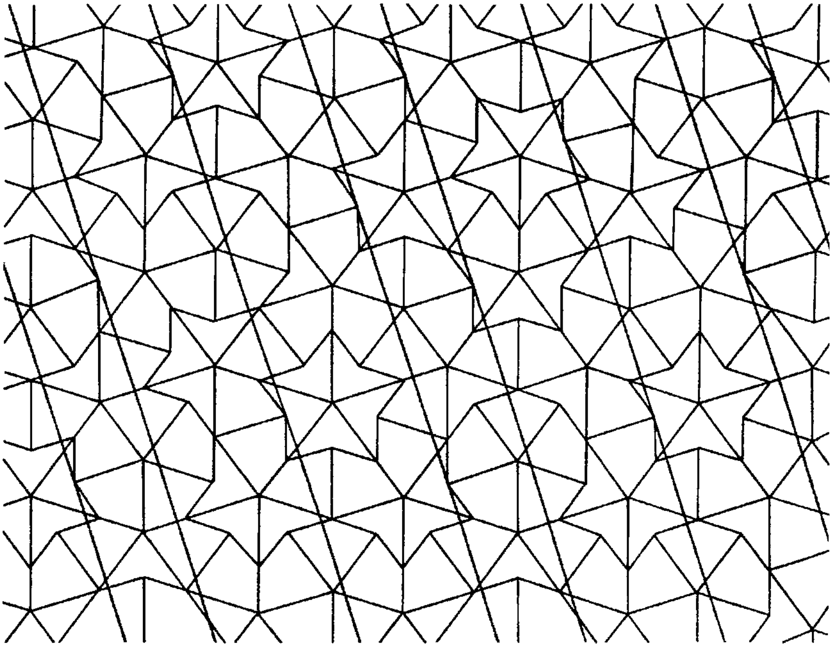
Begin on F_3, represented here by a whole note, and generate new pitches related by perfect fifths, keeping strictly within the octave between F_3 and F_4. (The assumption of Pythagorean tuning means that it is always possible to remain strictly within this octave.) The sequence of upward fifths and downward fourths corresponds to the sequence of *a*s and *b*s, respectively, in the binary sequence. The beginning of this sequence can be thought of as a tuning procedure for the diatonic scale; indeed, this is the procedure for tuning harps in some of the oldest musical evidence we possess, the Assyrian-Babylonian cuneiform tablets described by Kilmer, Crocker, and Brown (1976).

The notion of a scale with an infinite number of pitch classes confined within a finite octave is highly problematic. Possible realizations of a binary sequence in the rhythmic domain are conceivable, however, where the *a*s and *b*s are variously distinguished as short and long, or accented and unaccented, et cetera. Rhythmic interpretations of pitch structures have served as an analytical tool in Wooldridge 1992, and in the examinations of asymmetrical ostinatos in Rahn 1987 and Pressing 1993, and in Canright's study of Fibonacci rhythmic patterns (Canright 1992).

In Clough and Myerson 1985, scales with *MP* were constructed by taking integral parts of a rational number. The infinite binary sequences generated here by an irrational number still have *MP*: strings of *a*s and *b*s of any given length come in precisely two varieties. Moreover, the sequence of *a*s and *b*s is *quasi-periodic*, i.e., it never repeats itself, but every finite string of *a*s and *b*s repeats itself infinitely many times. Thus, the sequence is self-similar, indeed fractal in nature. Taking any segment of length n and calling the number of *a*s in it $\#a$, $\frac{\#a}{n}$ is a best approximation with denominator n to $\{\theta\}$ or $1-\{\theta\}$.

In the case of the *golden number*, the binary sequence is a one-dimensional analogue of a Penrose tiling, discussed in Grünbaum and Shephard 1986. A Penrose tiling is a two-dimensional nonperiodic tiling of the plane using two different shapes, known as kites and darts. A portion of a Penrose tiling is shown in Example 8. A Penrose tiling gives rise to five sets of parallel lines, called *Ammann bars*. One set of Ammann bars is shown in Example 8. The sequence of fat and thin Ammann bars is precisely the infinite binary sequence of *a*s and *b*s obtained by taking the integral parts of multiples of the golden number. Curiously, the mathematician John Conway refers to these as *musical sequences*.

A particular class of irrational numbers is comprised of the quadratic surds, that is, all irrational numbers x which satisfy equations $ax^2 + bx + c = 0$, where coefficients a, b, and c are integers, and a is positive. In general there are two solutions x which satisfy any given equation. We are concerned here only with positive values. Binary sequences derived from quadratic surds manifest another dimension of self-similarity. Example 9 displays part of the binary sequence determined by $\sqrt{2}$. With the help of a colleague at the Eastman School of Music, James Croson, we have produced a taped rhythmic realization of a portion of this binary sequence that articulates this other dimension of self-similarity.[6] The second line of numbers represents the integral parts of multiples of $\sqrt{2}$. The binary sequence is shown as Level 1. When the difference between two adjacent integral parts is 1 a small a appears on Level 1, and when the difference is 2, a b appears. On the tape, *b*s and *a*s were distinguished as accented and unaccented elements, respectively.



EXAMPLE 8

Again, the binary sequence is not periodic: it contains no finite subunit whose continued repetitions constitute the sequence as a whole. This is guaranteed by the fact that the ratio of #*b*s over any segment of length *n* to *n* approximates the irrational value $\{\sqrt{2}\}$. By element 46, there have been 19 *b*s, and $19/46 = 0.413\ldots$, a good approximation to $\{\sqrt{2}\}$.

As in all such sequences, one of the elements (here *b*) occurs less often than the other. Furthermore, the *b*s cannot occur twice in succession, and so Level 1 is partitioned into subunits of 2 or 3 elements, each subunit concluding with a *b*. Level 2 symbolizes this partitioning, where the larger subunit is labeled upper-case *B*, the smaller upper-case *A*. In this sequence as well, the *B*s occur only singly, and partition Level 2 into two- or three-element subunits, now with each subunit beginning on a *B*.

n: 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55

$[n\sqrt{2}]$: 0 1 2 4 5 7 8 9 11 12 14 15 16 18 19 21 22 24 25 26 28 29 31 32 33 35 36 38 39 41 42 43 45 46 48 49 50 52 53 55 56 57 59 60 62 63 65 66 67 69 70 72 73 74 76 77

1	*a a b*	*a b*	*a a b*	*a b*	*a a b*	*a b*	*a b*	*a a b*	*a b*	*a a b*	*a b*	*a b*	*a a b*	*a b*	*a a b*	*a b*	*a a b*	*a b*	*a b*	*a a b*	*a b*	*a a b*	*a*
2	*B*	*A*	*B*	*A*	*B*	*A*	*A*	*B*	*A*	*B*	*A*	*A*	*B*	*A*	*B*	*A*	*B*	*A*	*A*	*B*	*A*	*B*	
3	*a*		*a*		*b*			*a*		*b*			*a*		*a*		*b*			*a*			
4	*B*							*A*					*B*										

$[n\sqrt{2}] - [(n-1)\sqrt{2}] = 1 \rightarrow a \quad a\ b = A \qquad B\ A = a \qquad a\ b = A$

$[n\sqrt{2}] - [(n-1)\sqrt{2}] = 2 \rightarrow b \quad a\ a\ b = B \qquad B\ A\ A = b \qquad a\ a\ b = B$

$[x]$ = largest integer less than or equal to x

EXAMPLE 9: RHYTHMIC SEQUENCE GENERATED BY $\sqrt{2} = 1.4142135\ldots = [1, 2, 2, 2, \ldots]$

Level 3 in turn symbolizes this partitioning, with the larger *a* standing for each two-element subunit of Level 2 and *b* for each three-element subunit. Level 4 performs the same action on Level 3. Note that Level 3 is the same pattern as Level 1, and Level 4 reproduces Level 2. This procedure can obviously be extended indefinitely: each odd-numbered sequence is a reproduction of Level 1 and each even-numbered one reproduces Level 2. In fact, even- and odd-numbered sequences are, in a sense, retrogrades, which can be demonstrated by considering negative values for *n*.

Because the binary sequence is reproduced in augmentation at higher levels, if an accelerando is introduced, and elements are dynamically phased in and out, the pattern will return to its original state. This is a possible rhythmic analogue of the pitch phenomenon known as *Shepard tones*.[7] We attempted to implement this Shepard tone effect on our taped rhythmic realization.

* * *

The high degree of regularity produced in the binary sequence determined by $\sqrt{2}$ is a reflection of the periodicity of its continued fraction representation, a property shared by all quadratic surds. Continued fractions are also intimately linked to well-formed scales and their duals. The basic definitions concerning continued fractions are presented below. For more detailed information, see Hardy and Wright 1979, Chapter 10.

Every real number has a representation as a continued fraction. A rational number a/b is represented by a finite continued fraction, that is, a number of the form

$$t_0 + \cfrac{1}{t_1 + \cfrac{1}{t_2 + \cfrac{\ddots}{\ddots + \cfrac{1}{t_N}}}}$$

where t_0 is an integer, and the terms t_i are positive integers for all $i > 0$. The compact notation $[t_0, t_1, \ldots t_N]$ is used for convenience. An irrational number has a unique representation as an *infinite* continued fraction.

The rational numbers $[t_0], [t_0, t_1], [t_0, t_1, t_2], \ldots [t_0, t_1, \ldots t_k]$ are called the *convergents* of the continued fraction. In the case of an irrational number $\theta = [t_0, t_1, t_2, \ldots]$, the convergents approach θ as a limit. That is, we can approximate θ as closely as we wish by a convergent $[t_0, t_1, \ldots, t_k]$ with sufficiently many terms. The convergents are the best approximations to θ in the sense that if a/b is a convergent in lowest terms, there is no rational number with denominator less than or equal to *b* that is closer to θ. Finally, if $[t_0, t_1, \ldots, t_k]$ is a convergent where $t_k > 1$, we can define *semi-convergents* to be the numbers

$$[t_0, t_1, \ldots, t_{k-1}, 1], [t_0, t_1, \ldots, t_{k-1}, 2], \ldots, [t_0, t_1, \ldots, t_{k-1}, t_k - 1].$$

Convergents and semi-convergents are called odd or even, according to whether *k* is odd or even. A semi-convergent may be thought of as the best approximation to θ from one side, that is, an even semi-convergent is the best approximation from below and an odd semi-convergent is the best approximation from above.

All quadratic surds have continued fractions in which the terms repeat after a certain point, i.e., for all sufficiently large integers *k*, there exists some positive integer *i* such that $t_k = t_{k+i}$. Such continued fractions are called *periodic*, and a continued fraction is periodic if and only if it represents a quadratic surd. In the case of the continued fraction of $\sqrt{2}$, every term after the first is 2: $[1, 2, 2, 2, \ldots]$. In the case of

$$\phi = \frac{\sqrt{5}+1}{2},$$

the golden number, discussed above in connection with Penrose tilings, the continued fraction is the simplest of all: $[1, 1, 1, \ldots]$.

Recalling a result from Carey and Clampitt 1989, a generating interval of size θ determines a hierarchy of well-formed scales whose cardinalities are the denominators of the sequence of convergents and semi-convergents to θ. The hierarchy thus contains a finite number of scales when θ is rational, and an infinite number when irrational. For example, if $\theta = \log_2 3/2$, its continued fraction begins $[0, 1, 1, 2, 2, 3, 1, 5, 2, 23, \ldots]$, and the "best" approximations to θ are $0/1$, $1/1$, $1/2$, $2/3$, $3/5$, $4/7$, $7/12$, $10/17$, and so on, forming the sequence of convergents and semi-convergents to $\log_2 3/2$. The sequence is infinite, because $\log_2 3/2$ is an irrational number. The cardinalities of the well-formed scales in the hierarchy are therefore the denominators (1), 2, 3, 5, 7, 12, 17, . . . ad infinitum. The scale of cardinality 12 is the chromatic scale in Pythagorean tuning, with 7 steps having the ratio $256/243$ and 5 having the ratio $2187/2048$.

Each of these scales has an associated pattern, and the finite or infinite set of these patterns will be referred to as the scale pattern hierarchy. The $\log_2 3/2$ scale pattern hierarchy begins <*ab*>, <*baa*>, <*babaa*>, <*aaabaab*>, <*aababababaabab*>, et cetera. Since each of these patterns has a dual, we can consider the infinite set of these duals to be the dual hierarchy. See Example 10.

To recapitulate: a real number θ has associated with it a hierarchy of well-formed scales. This hierarchy in turn gives rise to a hierarchy of scale patterns. The duals of these scale patterns make up the dual hierarchy.

How is the dual hierarchy related mathematically to the original pattern hierarchy? As stated previously, the well-formed scales generated by θ are sets of fractional parts of multiples of θ and the cardinalities of these sets are determined by the continued fraction of θ. It turns out that the dual hierarchy is embodied in the binary sequence which is determined by *integral* parts of multiples of θ.

For example, let us consider the binary sequence of $\log_2 3/2$. Example 11 reproduces Example 6 with asterisks placed at certain points below the binary sequence. Comparing Examples 10 and 11 shows that the dual of each scale pattern of cardinality *n* in the hierarchy of $\log_2 3/2$ is a subsequence of length *n* in the binary sequence. These sequences extend from the origin to an asterisk. One finds the dual in a positive direction (to the right from the origin) if the associated convergent or semi-convergent is even, in a negative direction (to the left from the origin) if the associated convergent or semi-convergent is odd. As Example 10 shows, a number of the scale patterns in the hierarchy of $\log_2 3/2$ are self-dual. As it happens, up to this point the only scale patterns which are not self-dual are those with cardinalities 7 and 17.

The most redundant structures arise from the golden number. The pattern hierarchy and the dual hierarchy associated with the golden number are identical, because each scale pattern is its own dual. The pattern hierarchy associated with ϕ is the only self-dual hierarchy. Thus one obtains the same results whether one takes fractional or integral parts. Because the continued fraction of ϕ is $[1, 1, 1, \ldots]$, there are no semi-convergents, only full convergents. The cardinalities of the scales as well as the entries in the associated 2×2 arrays are all Fibonacci numbers.

If we group elements of the binary sequence of ϕ to obtain higher level sequences, as we did in the case of $\sqrt{2}$, the sequences are all identical augmentations of the original binary sequence. Canright (1992) suggests exploiting the ultimate degree of regularity available in this dimension of self-similarity.

* * *

	Pattern Hierarchy		Dual Hierarchy	
2	1 1 1 1	<*ab*>	1 1 1 1	<*ab*>
3	1 2 1 2	<*baa*>	1 2 1 2	<*bba*>
5	2 3 3 2	<*babaa*>	3 2 2 3	<*bbaba*>
7	2 5 4 3	<*aaabaab*>	4 3 2 5	<*abababb*>
12	5 7 5 7	<*aabababaabab*>	5 7 5 7	<*abababbababb*>
17	5 12 7 10	<*baabaabaaababaaa*>	7 10 5 12	<*bbababbababbababa*>
29	12 17 17 12	<*aababaababaabababaababaababab*>	17 12 12 17	<*bbababbababbabababbababbababa*>

EXAMPLE 10: THE PATTERN HIERARCHY AND THE DUAL HIERARCHY ASSOCIATED WITH $\log_2 \frac{3}{2}$

n:	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10	11	12
$\left[n \log_2 \frac{3}{2}\right]$:	–3	–3	–2	–2	–1	0	0	1	1	2	2	3	4	4	5	5	6	7
binary sequence:	*a*	*b*	*a*	*b*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*b*	*a*	*b*	*a*	*b*	*b*	
	*		*				*					*					*	

EXAMPLE 11: THE DUAL HIERARCHY EMBODIED IN THE BINARY SEQUENCE OF $\log_2 \frac{3}{2}$

Any realization of the binary sequence, no matter what the value of the generator, partakes of the self similarity that we first encountered in the diatonic scale. This self similarity makes the well-formed scale a very efficient information system, assuring that any sample reveals as much information as possible about the status of the whole, and making for an attractively complex system in which it is nevertheless easy to navigate, and an easy one to learn and remember.

Notes

1. An equivalent class of scales, called Moments of Symmetry, or MOS, was defined by Ervin Wilson in a private communication to John Chalmers in 1964, and was discussed in Chalmers 1975.

2. Hucbald shows the decomposition of nine intervals from the minor second through major sixth as follows: "Thus whereas the first interval is adjacent to equality itself, which it succeeds in the order of musical intervals, it appears when doubled to produce the second interval. But this is not precisely true, as will be discussed later. Therefore this second interval is regarded as being a certain amount more than the first one. The third interval consists of the first plus the second, that is, it has in itself as great a span as these together; the fourth interval consists of two of the second; the fifth, of the first plus two of the second; the sixth, of three of the second; the seventh, of the first and three of the second; the eighth, of four of the second; and the ninth, of the first and four of the second." (Babb 1978, 18.)

Despite his apparent awareness of the incommensurability of tones and semitones in Pythagorean tuning, he nonetheless parses the minor sixth as equal to four major seconds.

Prosdocimo, on the other hand, is particularly lucid here, providing an account of consonant intervals in full accord with Example 3, and notes the existence of what we have just labeled Myhill's Property: "It must be known, too, that any consonance save the unison, as well as every dissonance, is found in two inflections, the major and the minor. . . . [With regard to consonances t]he major third, then is that which contains two whole tones, the minor third is that which contains a whole tone with a semitone. The major fifth is that which contains three whole tones with one semitone and is that fifth which is counted among the perfect consonances, and which musical authors define as consisting of the sesquialtera proportion; the minor fifth is that which contains two whole tones with two semitones and is not a consonant interval but is counted among the intervals that are truly discordant. The major sixth is that which contains four whole tones with one semitone, the minor sixth that which contains three whole tones with two semitones. The major octave is that which contains five whole tones and two semitones. . . ." (Herlinger 1984, 45–47.)

Further, he writes: "every major interval found between the unison and the fifth interval (excluding both of these) contains no semitone; the minor interval is found to contain only a single semitone. Every major interval found between the fifth and the octave (including the fifth but excluding the octave) contains only a single semitone; the minor interval is found to contain two semitones." [p. 53.] Prosdocimo's observations are verified in columns 3 and 6 (no. of m2s) in Example 3.

3. Prosdocimo was aware of this invariance in the diatonic system, finding it important enough to be listed among his four intervallic rules: "The third rule is this; that any major interval exceeds the same minor interval by a major semitone; from which it follows that in reducing a major interval to its minor form, or vice versa, it is necessary only to add or subtract a major semitone." (Herlinger 1984, 55.) Given Prosdocimo's pro-Pythagorean, anti-Marchetto stance, "major semitone," signifies *apotomē* which indeed is the difference between the whole tone and the diatonic semitone (Prosdocimo's "minor semitone"). Equation 1.3 asserts that there will be such an invariance in every *MP* scale.

4. This scale is a "neutral third" scale: The fifth mode of this scale has the following values: (0, 200, 350, 500, 700, 850, 1050). If we associate the lowest pitch with the note C, then there are three major / minor triads: on C (0, 350, 700), F (500, 850, 1200) and G (700, 1050, 1400). Despite the fact that it seems possible to "derive" this scale from the diatonic, it is in fact of a radically different nature, as is apparent from the pattern of steps.

   An interesting relationship exists between the neutral third scale and Hauptmann's major and minor modes in Just Intonation. The two modes are arranged in thirds, originating on the same pitch:

F	a	C	e	G	b	D
F	a♭	C	e♭	G	b♭	D

The notes are put into scale order starting on C:

	C	D	E	F	G	A	B	C
Major	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$

	C	D	E♭	F	G	A♭	B♭	C
minor	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	$\frac{2}{1}$

Geometric means are taken between comparable notes in both scales to form a composite scale:

neutral	$\frac{1}{1}$	$\frac{9}{8}$	$\sqrt{\frac{3}{2}}$	$\frac{4}{3}$	$\frac{3}{2}$	$\sqrt{\frac{8}{3}}$	$\sqrt{\frac{27}{8}}$	$\frac{2}{1}$

The basic pattern of this mode <*abbabab*> is realized here with the ratio $\frac{9}{8}$ as the *a* step, and $\sqrt{\frac{32}{27}}$ as the *b*. Note that the values here are, of course, frequency ratios, not logarithms. The actual logarithms are all within four cents of those given at the beginning of this note.

5. Another view of the role of the triad in the well-formed diatonic scale is presented in Carey and Clampitt 1989. See pages 203–5. The discussion there concerns the three distinct automorphisms of a seven note set, which give rise to the three generic interval cycles: steps (or sevenths), thirds (or sixths), and fifths (or fourths). The cycle of thirds (Major and minor) is some rotation of <MmMmMmm>, the dual of the diatonic. There is also some analogy here to the notion of "second order" maximally even sets in Clough and Douthett 1991. Clough and Douthett propose that the triad is second order maximally even with respect to the diatonic.

6. We presented two versions of this realization at the 1995 annual conference of the Society for Music Theory in New York City. Croson has subsequently used this binary sequence as the basis for a more extended electronic composition.

7. See Shepard (1964). For another type of rhythmic realization of the Shepard tone paradox, see Jean-Claude Risset (1989).

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